Inverse magnetisation problem for paleomagnetism: reconstruction of net magnetisation through asymptotic analysis and field extrapolation

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Abstract

Motivated by a concrete experimental setting in the Paleomagnetism lab at EAPS department of MIT, we consider a particular instance of the inverse magnetisation problem. If the measurement area was large, a set of explicit asymptotic estimates could be used to estimate the net magnetisation of a sample. In particular, new high-order estimates could be used to mitigate the large-measurement-area assumption. However, these formulae are less stable with respect to noise which obstructs their direct use in practice. As a remedy, when the magnetisation distribution is planar and $W^{1,2}$-regular, we propose a method to extrapolate the measurements to a larger area and denoise them at the same time.

Keywords: paleomagnetism, inverse source problems, ill-posed problems, extrapolation

1 Introduction

The process of extraction of relict magnetic information from georocks and meteorites is a challenging but important task in paleomagnetic research. Due to the weak intensity of the field produced by a magnetised rock, the measurements have to be performed in the direct vicinity of the sample and using highly sensitive magnetometric devices such as SQUID (superconducting quantum interference device) and QDM (quantum diamond microscope). The basic quantity of interest is the net magnetisation (magnetisation moment vector). Reconstruction of this quantity hinges on effective processing of the experimental data, with the main challenges being the limited measurement area and the noise contamination.

In an experimental setup at EAPS, MIT, only the component B_3 of the magnetic field $\vec{B}$ is measured on a portion of the orthogonal plane at distance h above a sample with unknown magnetisation distribution $\vec{\mathcal{M}} \equiv (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)^T$ supported on a compact set $Q \subset \mathbb{R}^3$. These quantities are related as

$$B_3(\mathbf{x}, h) = \frac{1}{4\pi}\frac{\partial}{\partial h}\int_Q \frac{\nabla \cdot \vec{\mathcal{M}}(\mathbf{t}, t_3)\, \mathrm{d}^3 t}{\left(|\mathbf{x}-\mathbf{t}|^2 + (h-t_3)^2\right)^{1/2}}, \tag{1}$$

where we used the notation $\mathbf{x} \equiv (x_1, x_2)^T$. The problem of reconstuction of $\vec{\mathcal{M}}$ from B_3 is known to be severely ill-posed [1]. However, a quantity of principal interest that (at least in theory) can be uniquely defined from B_3 is the net magnetisation vector

$$\vec{m} \equiv (m_1, m_2, m_3)^T := \int_Q \vec{\mathcal{M}}(\vec{x})\, \mathrm{d}^3 x.$$

2 Some asymptotic results

When the measurement area D_A is the disk of a large radius A, the estimates of components of the net magnetisation vector are known [2]:

$$m_{1,2} = 2\int_{D_A} x_{1,2} B_3(\mathbf{x}, h)\, \mathrm{d}^2\mathbf{x} + \mathcal{O}\left(\frac{1}{A}\right),$$

$$m_3 = 2A\int_{D_A} B_3(\mathbf{x}, h)\, \mathrm{d}^2\mathbf{x} + \mathcal{O}\left(\frac{1}{A^2}\right).$$

One can also obtain and rigorously prove analogous asymptotic estimates of higher orders [3]. For example, a 4th-order formula for m_3 reads:

$$m_3 = \frac{A}{24}\int_{D_A}\left[35 + 1792\left(\frac{x_j}{A}\right)^6 - 3200\left(\frac{x_j}{A}\right)^8\right] B_3(\mathbf{x}, h)\, \mathrm{d}^2\mathbf{x} + \mathcal{O}\left(\frac{1}{A^4}\right),$$

where x_j is either x_1 or x_2.

3 Field extrapolation method

Even though they are destined to be beneficial for a smaller measurement region, high-order asymptotic formulas are much more unstable with respect to noise. In order to improve their robustness, we propose to extrapolate the field in the measurement plane while mitigating the effects of the noise. Here, we assume that magnetisation is planar, i.e. $\vec{\mathcal{M}}(\vec{x}) = \vec{M}(\mathbf{x})\otimes\delta(x_3)$, where δ is the Dirac's delta, $\vec{M} \in \left[W^{1,2}\left(\mathbb{R}^2\right)\right]^3$, supp $\vec{M} \subset Q_0 \subset \mathbb{R}^2$ and Q_0 is a bounded region in the measurement plane above the sample where the data are available. Note that the assumption on the magnetisation being planar is consistent with the experimental setup where a magnetic rock is sliced into a thin slab.

In order to obtain the extrapolated field B_3^{ext} defined on $\widetilde{Q}_0$ from the measured field B_3^{meas} defined on Q_0, we propose the following strategy.

Step 1: Construct a special set of basis functions $(\varphi_n)_{n=1}^{\infty} \subset L^2(Q_0)$ related to integral operators of the direct problem (1). More precisely, φ_n are solutions (for appropriate eigenvalues λ_n) of the integral equation

$$K_1 \star \chi_{Q_0}\varphi_n + K_2 \star \chi_{Q_0}\mathcal{R}[\varphi_n] = \lambda_n\varphi_n \quad \text{on } Q_0, \tag{2}$$

where χ_{Q_0} is the characteristic function of Q_0,

$$K_j(\mathbf{x}) := \frac{\partial^{j-1}}{\partial h^{j-1}} \frac{-h}{4\pi\left(|\mathbf{x}|^2 + h^2\right)^{3/2}}, \quad j \in \{1, 2\},$$

$$\mathcal{R}[f](\mathbf{x}) := \int_{Q_0} R(\mathbf{x}, \mathbf{t})\, f(\mathbf{t})\, \mathrm{d}^2\mathbf{t}, \quad f \in L^2(Q_0),$$

with $R(\mathbf{x}, \mathbf{t})$ *approximately* satisfying

$$\int_{Q_0} K_1(\mathbf{y} - \mathbf{t})\, R(\mathbf{x}, \mathbf{t})\, \mathrm{d}^2\mathbf{t} = K_2(\mathbf{x} - \mathbf{y}),$$

for any $\mathbf{x}$, $\mathbf{y} \in Q_0$.

Step 2: Approximate the measured field by the expansion

$$B_3^{\text{meas}}(\mathbf{x}) \simeq \sum_{n=1}^{N} b_n\varphi_n(\mathbf{x}), \quad \mathbf{x} \in Q_0,$$

$$b_n = \left\langle B_3^{\text{meas}}, \varphi_n + \mathcal{R}^\star\mathcal{R}[\varphi_n]\right\rangle_{L^2(Q_0)}, \tag{3}$$

where $\mathcal{R}^\star$ is the operator adjoint to $\mathcal{R}$ and the number N is chosen based on a compromise between the approximation quality and the data fidelity (noise level in B_3^{meas}).

Step 3: Compute the extrapolant:

$$B_3^{\text{ext}}(\mathbf{x}) = \sum_{n=1}^{N} b_n\widetilde{\varphi}_n(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2,$$

where b_n are as in (3), and the functions

$$\widetilde{\varphi}_n := \frac{1}{\lambda_n}\left(K_1 \star \chi_{Q_0}\varphi_n + K_2 \star \chi_{Q_0}\mathcal{R}[\varphi_n]\right),$$

are all defined on $\mathbb{R}^2$ as are K_1, K_2 since φ_n on Q_0 are already found from (2). Note that we obviously have $\widetilde{\varphi}_n|_{Q_0} = \varphi_n$.

Figure 1 provides numerical illustration of the extrapolation algorithm with denoising.

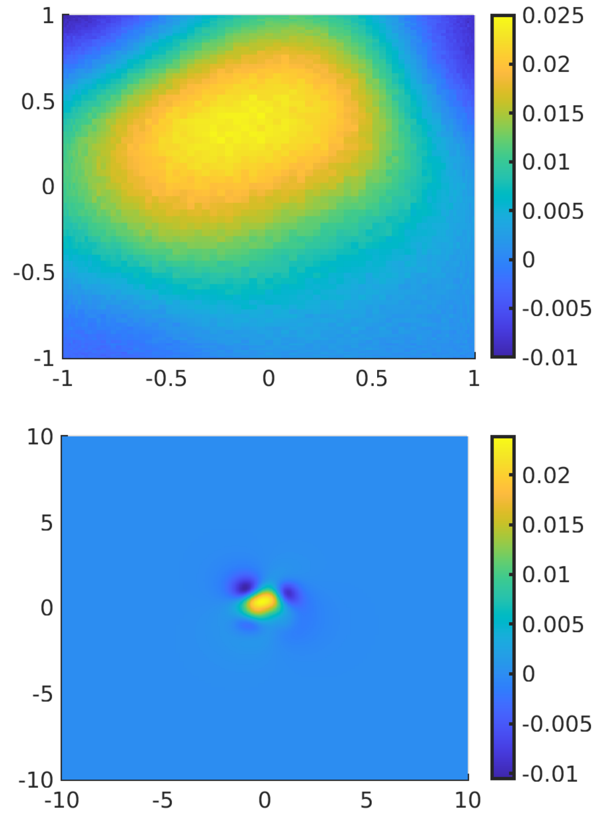

Figure 1: Original field B_3^{meas} contaminated with 5% of additive Gaussian white noise [top] and B_3^{ext}, its denoised version extrapolated to a 100 times larger area [bottom].

References

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