JUSTIFICATION OF A NONLINEAR SCHRÖDINGER MODEL FOR POLYMERS

By

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Abstract

A model with nonlinear Schrödinger (NLS) equation used for describing pulse propagations in photopolymers is considered. We focus on a case in which change of refractive index is proportional to the square of amplitude of the electric field and the spatial domain is $\mathbb{R}^2$. After formal derivation of the NLS approximation from the wave-Maxwell equation, we establish well-posedness and perform rigorous justification analysis to show smallness of error terms for appropriately small time intervals. We conclude by numerical simulation to illustrate the results in one-dimensional case.

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To my parents

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Contents

Chapter 1

Introduction

1.1 Physical context

Mathematical models for laser beams in photochemical materials used in literature [8, 16] are based on the nonlinear Schrödinger (NLS) equation. These models are normally derived from Maxwell equations using heuristic arguments and qualitative approximations (see e.g. [9]). In the present work we derive the time-dependent NLS equation rigorously from a toy model resembling the Maxwell equations. The toy model is written as a system of a wave-Maxwell equation and an empirical relation for the change of the refractive index.

The following planar geometry problem is usually considered for modeling of laser beams in photochemical materials. The material occupies halfspace $z \geq 0$ and its face $z = 0$ is exposed to the pulse entering the material. If the pulse is localized in the x-direction and is uniform in the y-direction, then the electric field has polarization in the y-direction with the amplitude E being y-independent, hence $\mathbf{E}(x, z, t) = (0, E(x, z, t), 0)$ is the electric field. The initial pulse is assumed to be spatially wide-spreaded, small in amplitude, and monochromatic in time.

Neglecting polarization effects and uniform material losses, we write the wave-Maxwell equation in the form

$$\partial_x^2 E + \partial_z^2 E - n^2 \partial_t^2 E = 0, \tag{1.1}$$

where n is referred to as the refractive index of the photochemical material.

Let us write the squared refractive index in the form $n^2 = 1 + m$ and assume that the change of refractive index m is governed by the empirical relation

$$\frac{\partial m}{\partial t} = E^2. \tag{1.2}$$

We note that all physical constants are normalized to be 1 in the system (1.1)-(1.2).

The system (1.1)-(1.2) resembles a more complicated system of governing equations in literature [8].

1.2 Asymptotic balance

Let us seek for the asymptotic solution to the system (1.1)-(1.2) by using the multi-scale expansion [10, 14, 17]

$$E(x,z,t) = \epsilon^p A(X,Z,T)\, e^{i\omega_0(z-t)} + c.c., \quad m(x,z,t) = \epsilon^r m_0(X,Z,T), \tag{1.3}$$

where $c.c.$ stands for complex conjugated term, $X := \epsilon x$, $Z := \epsilon^q z$, $T := \epsilon^s t$ are slow variables and p, q, s, $r > 0$ are exponents to be specified.

We want to choose the exponents p, q, s and r such that A is governed by a kind of the NLS equation.

The resulting NLS equation for A is supposed to have first-order partial derivatives of A in Z, second derivative in X, and a nonlinear term proportional to $m_0 A$ at the leading order of ϵ (that is $\mathcal{O}\left(\epsilon^{p+2}\right)$ due to the term $\partial_x^2 E$). At the same time the equation (1.2) must enforce the rate of change of m_0 in T to be of order $\mathcal{O}(1)$ at the leading order of ϵ (that is $\mathcal{O}\left(\epsilon^{2p}\right)$ due to the term E^2). These requirements lead to the choice

$$q = 2, \quad r = 2, \quad s = 2p - 2, \tag{1.4}$$

which still leaves parameter p to be defined.

To show (1.4), we substitute (1.3) in (1.1) and (1.2) to obtain, respectively,

$$\epsilon^p \left[\epsilon^2 \partial_X^2 A + 2i\omega_0 \left(\epsilon^q \partial_Z A + \epsilon^s \partial_T A\right) + \epsilon^r \omega_0^2 m_0 A\right] e^{i\omega_0(z-t)} + c.c. + \text{higher-order terms} = 0,$$

$$\epsilon^{r+s} \partial_T m_0 = \epsilon^{2p} |A|^2 + \left(\epsilon^{2p} A^2 e^{2i\omega_0(z-t)} + c.c.\right) + \text{higher-order terms}.$$

From the first equation, the balance occurs for $q = 2$, $r = 2$ and $s \geq 2$. From the second equation, the balance occurs for $r + s = 2p$, hence $s = 2p - 2$, and the balance (1.4) is justified.

The second term in the second equation induces the second harmonics which will be further included in the equation for a residual term.

If $s = 2$, the system of equations can be truncated at the system

$$\partial_X^2 A + 2i\omega_0 \left(\partial_Z A + \partial_T A\right) + \omega_0^2 m_0 A = 0, \tag{1.5}$$

$$\partial_T m_0 = 2 \left|A\right|^2. \tag{1.6}$$

If $s > 2$, the system of equations can be truncated at the spatial NLS equation

$$\partial_X^2 A + 2i\omega_0 \partial_Z A + \omega_0^2 m_0 A = 0, \tag{1.7}$$

complimented by the same equation (1.6). Because m_0 depends on T by means of the equation (1.6), A depends on T implicitly in the case of (1.7). The system (1.6)-(1.7) was used in the previous works [8, 16] on photochemical materials.

Our task is to justify the system (1.5)-(1.6), where the time evolution of A is uniquely determined. To avoid problems at the characteristics $Z = T$, we shall consider solutions of the original system (1.1)-(1.2) in an unbounded domain $(x, z) \in \mathbb{R}^2$ for $t > 0$ supplemented by the initial conditions. At the present time, our method does not allow us to justify the system (1.6)-(1.7).

In the case $s = 2$, we choose the scaling $X := \epsilon x$, $Z := \epsilon^2 z$, $T := \epsilon^2 t$ and represent the exact solution to (1.1)-(1.2) as

$$E\left(x, z, t\right) = \epsilon^2 \left(A\left(X, Z, T\right) e^{i\omega_0 (z-t)} + c.c.\right) + U\left(x, z, t\right), \tag{1.8}$$

$$m\left(x, z, t\right) = \epsilon^2 m_0 \left(X, Z, T\right) + N\left(x, z, t\right), \tag{1.9}$$

where $U(x,z,t)$, $N(x,z,t)$ are residual terms to estimate.

Let us denote

$$(X)_{n\omega_0} := Xe^{in\omega_0(z-t)} + c.c.$$

for any complex envelope X of the n-th harmonic.

Feeding (1.8)-(1.9) into (1.1)-(1.2) and assuming validity of (1.5)-(1.6), we arrive at

$$\partial_x^2 U + \partial_z^2 U - \left(1+\epsilon^2 m_0 + N\right)\partial_t^2 U = -\epsilon^2 \left(R_2^{(U)}\right)_{\omega_0} N - \epsilon^6 \left(R_6^{(U)}\right)_{\omega_0}, \tag{1.10}$$

$$\partial_t N = \epsilon^4 \left(A^2\right)_{2\omega_0} + 2\epsilon^2 (A)_{\omega_0} U + U^2, \tag{1.11}$$

where

$$R_2^{(U)} := \omega_0^2 A + 2i\omega_0\epsilon^2 \partial_T A - \epsilon^4 \partial_T^2 A, \tag{1.12}$$

$$R_6^{(U)} := \partial_Z^2 A - \left(1+\epsilon^2 m_0\right)\partial_T^2 A + 2i\omega_0 m_0 \partial_T A. \tag{1.13}$$

1.3 Main result

For the system (1.1)-(1.2), we impose the following initial conditions

$$E|_{t=0} =: E_0 = \epsilon^2 A_0\left(\epsilon x, \epsilon^2 z\right) e^{i\omega_0 z} + c.c., \tag{1.14}$$

$$\partial_t E|_{t=0} =: E_1 = -i\omega_0\epsilon^2 A_0\left(\epsilon x, \epsilon^2 z\right) e^{i\omega_0 z} + \epsilon^4 \partial_T A_0\left(\epsilon x, \epsilon^2 z\right) e^{i\omega_0 z} + c.c., \tag{1.15}$$

$$m|_{t=0} = 0, \tag{1.16}$$

where A_0 is the initial distribution of the beam for the Schrödinger equation and $\partial_T A_0$ is expressed explicitly from (1.5). The initial conditions imply that at $t=0$ the electrical field is already penetrated in the photochemical material, whereas it does not yet induce the change in the refractive index. Note also that the conditions (1.14)-(1.16) imply that $U|_{t=0} = \partial_t U|_{t=0} = N|_{t=0} = 0$ in the system (1.10)-(1.11) for the residual terms.

Our main result is the following justification theorem.

Theorem 1.1. *Given initial data $A_0 \in H^8\left(\mathbb{R}^2\right)$, let A, m_0 be local solutions to the system (1.5)-(1.6) for $T \in (0, T_\infty)$ where $T_\infty > 0$ is the maximal existence time. There exist $\epsilon_0 > 0$ and $T_0 \in (0, T_\infty)$ such that for every $\epsilon \in (0, \epsilon_0)$ there is a unique solution E, m of the system (1.1)-(1.2) for $t \in \left[0, T_0/\epsilon^2\right]$*

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\| E - \epsilon^2 \left(A\right)_{\omega_0} \right\|_{H^3(\mathbb{R}^2)} = \mathcal{O}\left(\epsilon^{5/2}\right),$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\| m - \epsilon^2 m_0 \right\|_{H^2(\mathbb{R}^2)} = \mathcal{O}\left(\epsilon^{5/2}\right).$$

To prove the Theorem 1.1, we organize thesis as follows.

In the Chapter 2, we will set up tools of functional analysis needed for our work. After the brief introduction of notation, the main lemmas that will be used throughout the work are stated. The Chapter 3 describes the local well-posedness theory for the original system (1.1)-(1.2) and its approximation (1.5)-(1.6). We obtain the spaces for the local solution in which further analysis will be done, and formulate the theorem that allows continuation of the local solutions. Additionally, we look into the smoothness requirements of the initial data of the initial pulse. The goal of the Chapter 4 is to obtain sufficient estimates for the residual terms U, N governed by the equations (1.10)-(1.11) and hence to justify the approximation of solutions of the system (1.1)-(1.2) by solutions of the NLS system (1.5)-(1.6). This is done by means of normal form transformation followed by a priori energy estimates which yield control of the residual terms. The Chapter 5 illustrates numerically the result of the Theorem 1.1 for x-independent initial conditions. In the Appendix, we obtain global justification for the linear counterpart of the problem by comparing solutions of the original wave equation with solution to the linear Schrödinger equation.

Chapter 2

Elements of functional analysis

In this chapter we collect together some definitions and results from topics in functional analysis. These results will be used in the rest of our work.

For a positive integer s, $H^s\left(\mathbb{R}^2\right) := W^{s,2}\left(\mathbb{R}^2\right)$ denotes the Hilbert-Sobolev space, that is, the space of all functions of two variables bounded with respect to the induced norm

$$\|f\|_{H^s} := \left(\sum_{0\le k+l\le s} \int_{\mathbb{R}^2} \left|\partial_x^k \partial_z^l f\right|^2 dxdz \right)^{1/2},$$

or equivalently,

$$\|f\|_{H^s} = \left(\sum_{k+l=s} \int_{\mathbb{R}^2} \left|\partial_x^k \partial_z^l f\right|^2 dxdz \right)^{1/2} + \left(\int_{\mathbb{R}^2} |f|^2 \, dxdz \right)^{1/2}.$$

We will use standard notation for the Lebesgue spaces $L^p\left(\mathbb{R}^2\right)$ endowed with the norm

$$\|X\|_{L^p} := \left(\int_{\mathbb{R}^2} |f(x,z)|^p \, dxdz \right)^{1/p}, \quad 1 \le p < \infty.$$

In addition, we have

$$\|X\|_{L^\infty} := \lim_{p\to\infty} \|X\|_{L^p} = \operatorname*{ess\,sup}_{(x,z)\in\mathbb{R}^2} |f(x,z)|\,.$$

Now, let us assume that functions f in $H^s\left(\mathbb{R}^2\right)$ depend on an additional variable $t \in \mathbb{R}_+$. In what follows, we will often write $f \in H^s$ implying $f(\cdot,\cdot,t) \in H^s\left(\mathbb{R}^2\right)$.

Lemma 2.1. *Assume that f, $\partial_t f \in L^p\left(\mathbb{R}^2\right)$. Then for any $1 \le p < \infty$, we have*

$$\partial_t \|f\|_{L^p} \le \|\partial_t f\|_{L^p}. \tag{2.1}$$

Proof. Clearly,

$$\partial_t \|f\|_{L^p}^p = p\,\|f\|_{L^p}^{p-1}\,\partial_t \|f\|_{L^p}. \tag{2.2}$$

On the other hand, for $1 \le p < \infty$, by the Lebesgue dominated convergence theorem (valid since $f \in L^p$, $\partial_t f \in L^p$), differentiation can be performed under the integral sign which is then followed by application of the Hölder's inequality

$$\partial_t \|f\|_{L^p}^p = p\int_{\mathbb{R}^2} |f(x,z,t)|^{p-1}\,\partial_t f(x,z,t)\,dxdz \le p\left\|f^{p-1}\right\|_{L^{p/(p-1)}}\|\partial_t f\|_{L^p} = p\,\|f\|_{L^p}^{p-1}\|\partial_t f\|_{L^p}. \tag{2.3}$$

Comparison of (2.2) and (2.3) furnishes the result (2.1). □

Corollary 2.1. *Assume that f, $\partial_t f \in L^p\left(\mathbb{R}^2\right)$ for all $t \in [0,t_0]$ and some $p \ge 1$. Then, we have*

$$\|f\|_{L^p} \le t_0 \sup_{t\in[0,t_0]} \|\partial_t f\|_{L^p} + \left(\|f\|_{L^p}\right)\big|_{t=0}, \quad t \in [0,t_0]. \tag{2.4}$$

Proof. For $p < \infty$, this result follows directly from the Lemma 2.1.

For $p = \infty$, the results follows from the fundamental theorem of calculus and the integral Minkowski's inequality

$$\begin{aligned}\|f\|_{L^\infty} &\le \left\|\int_0^t \partial_\tau f d\tau\right\|_{L^\infty} + \left(\|f\|_{L^\infty}\right)\big|_{t=0}\\ &\le t_0 \sup_{t\in[0,t_0]} \|\partial_t f\|_{L^\infty} + \left(\|f\|_{L^\infty}\right)\big|_{t=0}, \quad t\in[0,t_0].\end{aligned}$$

□

Corollary 2.2. *Let f, $\partial_t f \in H^s\left(\mathbb{R}^2\right)$ for all $t \in [0,t_0]$ and some $s \ge 0$. Then we have*

$$\|f\|_{H^s} \le t_0 \sup_{t\in[0,t_0]} \|\partial_t f\|_{H^s} + \left(\|f\|_{H^s}\right)\big|_{t=0}, \quad t \in [0,t_0]. \tag{2.5}$$

Proof. By the Plancherel's theorem (see e.g. [12, 13]), we can employ the estimate (2.4) for $p = 2$ on the Fourier transform side $f(\boldsymbol{\xi}) \in L^2(\mathbb{R}^2)$

$$\begin{aligned}
\|f\|_{H^s} &= \left\| \left(1 + |\boldsymbol{\xi}|^2\right)^{s/2} \hat{f} \right\|_{L^2} \\
&\leq t_0 \sup_{t\in[0,t_0]} \left\| \left(1 + |\boldsymbol{\xi}|^2\right)^{s/2} \partial_t \hat{f} \right\|_{L^2} + \left(\left\| \left(1 + |\boldsymbol{\xi}|^2\right)^{s/2} \hat{f} \right\|_{L^2} \right)\Big|_{t=0} \\
&\leq t_0 \sup_{t\in[0,t_0]} \|\partial_t f\|_{H^s} + \left(\|f\|_{H^s}\right)\big|_{t=0}.
\end{aligned}$$

□

In the rest of this section, we list useful results: Banach algebra property, Sobolev embedding theorem, Gagliardo-Nirenberg inequality and Gronwall's inequality, and Banach fixed-point theorem. For the proofs, see [1, 3, 10].

Proposition 2.1. *(Banach algebra property) For any $s > 1$, $H^s(\mathbb{R}^2)$ is a Banach algebra with respect to multiplication, that is, if $f, g \in H^s(\mathbb{R}^2)$, then there is a constant $C_s > 0$ (depending only on index s) such that*

$$\|fg\|_{H^s} \leq C_s \|f\|_{H^s} \|g\|_{H^s}. \tag{2.6}$$

Proposition 2.2. *(Sobolev embedding) Assume that $f \in H^s(\mathbb{R}^2)$ for $s \geq 2$. Then, the function f is continuous on $\mathbb{R}^2$ decaying at infinity, and there is a constant $C_s > 0$ such that*

$$\|f\|_{L^\infty} \leq C_s \|f\|_{H^s}. \tag{2.7}$$

Proposition 2.3. *(Gagliardo-Nirenberg inequality) Let $f \in H^1(\mathbb{R}^2)$. Then, for any $\sigma \geq 0$, there exists a constant $C_\sigma > 0$ such that*

$$\|f\|_{L^{2(\sigma+1)}}^{2(\sigma+1)} \leq C_\sigma \|\nabla f\|_{L^2}^{2\sigma} \|f\|_{L^2}^2. \tag{2.8}$$

Proposition 2.4. *(Gronwall's inequality) Assume* $g(t) \in C^1([0, t_0])$ *satisfies*

$$\frac{dg(t)}{dt} \leq ag(t) + b, \quad t \in [0, t_0].$$

for some constants a, $b > 0$ *and* $g(0) > 0$. *Then, we have*

$$g(t) \leq (g(0) + bt_0)\, e^{at}, \quad t \in [0, t_0]. \tag{2.9}$$

Proposition 2.5. *(Banach fixed-point theorem) Let* $\mathcal{B}$ *be a closed non-empty set of the Banach space* X, *and let* $K : \mathcal{B} \mapsto \mathcal{B}$ *be a contraction operator, that is, for any* x, $y \in \mathcal{B}$, *there exists* $0 \leq q < 1$ *such that* $\|K(x) - K(y)\|_X \leq q\, \|x - y\|_X$. *Then there exists a unique fixed point of* K *in* $\mathcal{B}$, *in other words, there exists a unique solution* $x_0 \in \mathcal{B}$ *such that* $K(x_0) = x_0$.

Chapter 3

Local well-posedness theory

3.1 Local well-posedness of the wave-Maxwell system

Before we proceed with the justification analysis, let us consider the question of local well-posedness of the system (1.1)-(1.2) and formulate a regularity criterion for the continuations of local solutions.

Consider the wave-Maxwell system

$$\begin{cases} \partial_x^2 E + \partial_z^2 E - (1+m)\,\partial_t^2 E = 0, \\ \partial_t m = E^2, \end{cases} \qquad (x,z) \in \mathbb{R}^2,\, t \in \mathbb{R}_+, \tag{3.1}$$

subject to the initial conditions $m|_{t=0} = 0$, $E|_{t=0} = E_0$, and $E_t|_{t=0} = E_1$ for given E_0, $E_1 \in H^s\left(\mathbb{R}^2\right)$ with some $s \geq 0$.

We can apply the theory of local well-posedness for quasi-linear symmetric hyperbolic systems [5, 7, 15] once we bring (3.1) into a first-order system with a symmetric matrix.

To symmetrize the system, we set

$$\mathbf{v} := \left(\partial_t E,\, \frac{\partial_x E}{(1+m)^{1/2}},\, \frac{\partial_z E}{(1+m)^{1/2}},\, E,\, \partial_x m,\, \partial_z m,\, m\right)^T. \tag{3.2}$$

Then, the system (3.1) is equivalent to the symmetric quasi-linear first-order system

$$\partial_t \mathbf{v} + A_1(\mathbf{v})\,\partial_x \mathbf{v} + A_2(\mathbf{v})\,\partial_z \mathbf{v} = \mathbf{f}(\mathbf{v})\,, \tag{3.3}$$

where A_1, A_2 are matrices having $-\frac{1}{(1+v_7)^{1/2}}$ at $(1,2)$-$(2,1)$ and $(1,3)$-$(3,1)$ entries, respectively, and zero elements elsewhere, whereas

$$\begin{aligned}\mathbf{f}(\mathbf{v}) &:= \left(\frac{v_2v_5+v_3v_6}{2(1+v_7)^{3/2}},\, -\frac{v_4^2v_2}{2(1+v_7)},\, -\frac{v_4^2v_3}{2(1+v_7)},\, v_1,\, 2(1+v_7)^{1/2}v_2v_4,\, 2(1+v_7)^{1/2}v_3v_4,\, v_4^2\right)^T \\ &= \left(\frac{\partial_xE\partial_xm+\partial_zE\partial_zm}{2(1+m)^2},\, -\frac{E^2\partial_xE}{2(1+m)^{3/2}},\, -\frac{E^2\partial_zE}{2(1+m)^{3/2}},\, \partial_tE,\, 2E\partial_xE,\, 2E\partial_zE,\, E^2\right)^T.\end{aligned}$$

Note that A_1, A_2, $\mathbf{f}$ have no explicit dependence on x, z and t.

The initial data for (3.3) is given by

$$\mathbf{v}|_{t=0} = \mathbf{v}_0 := (E_1,\, \partial_xE_0,\, \partial_zE_0,\, E_0,\, 0,\, 0,\, 0)^T. \tag{3.4}$$

By the Kato theory (see Theorems I-II in [5]), for any integer $s \geq 3$, the Cauchy problem (3.3)-(3.4) admits unique local solution $\mathbf{v} \in C\left([0,t_0], H^s\left(\mathbb{R}^2\right)\right) \cap C^1\left([0,t_0], H^{s-1}\left(\mathbb{R}^2\right)\right)$ for some $t_0 > 0$ providing $\mathbf{v}_0 \in H^s\left(\mathbb{R}^2\right)$. Moreover, the solution $\mathbf{v}$ depends on the initial data $\mathbf{v}_0$ continuously (Theorem III in [5]). We transfer this result in the following statement.

Lemma 3.1. *For any integer $s \geq 3$, the unique local solution of the system (3.1) exists in the space*

$$E \in C\left([0,t_0], H^{s+1}\left(\mathbb{R}^2\right)\right) \cap C^1\left([0,t_0], H^s\left(\mathbb{R}^2\right)\right) \cap C^2\left([0,t_0], H^{s-1}\left(\mathbb{R}^2\right)\right), \tag{3.5}$$

$$m \in C^1\left([0,t_0], H^{s+1}\left(\mathbb{R}^2\right)\right) \cap C^2\left([0,t_0], H^s\left(\mathbb{R}^2\right)\right) \cap C^3\left([0,t_0], H^{s-1}\left(\mathbb{R}^2\right)\right). \tag{3.6}$$

Moreover, the solution depends continuously on the initial data $E_0 \in H^{s+1}\left(\mathbb{R}^2\right)$, $E_1 \in H^s\left(\mathbb{R}^2\right)$.

Proof. From the first and the last four entries in (3.2), we infer that, for any integer $s \geq 3$,

$$E \in C^1\left([0,t_0], H^s\left(\mathbb{R}^2\right)\right) \cap C^2\left([0,t_0], H^{s-1}\left(\mathbb{R}^2\right)\right), \tag{3.7}$$

$$m \in C\left([0,t_0], H^{s+1}\left(\mathbb{R}^2\right)\right) \cap C^1\left([0,t_0], H^s\left(\mathbb{R}^2\right)\right). \tag{3.8}$$

We shall now use the second and third entries in (3.2), which tell us that

$$J := \int_{\mathbb{R}^2} \left(\left[\partial_x^s \left(\frac{\partial_x E}{(1+m)^{1/2}} \right) \right]^2 + \left[\partial_z^s \left(\frac{\partial_z E}{(1+m)^{1/2}} \right) \right]^2 \right) dxdz < \infty,$$

and J is a continuous function of t on$[0, t_0]$.

Without loss of generality, let us keep track of only x-derivatives.

By the Leibnitz differentiation rule, we have

$$\begin{aligned} \left[\partial_x^s \left(\frac{\partial_x E}{(1+m)^{1/2}} \right) \right]^2 &= \left[\sum_{k=0}^{s} \begin{pmatrix} s \\ k \end{pmatrix} \partial_x^{k+1} E \partial_x^{s-k} (1+m)^{-1/2} \right]^2 \\ &= \left[(1+m)^{-1/2} \partial_x^{s+1} E + \sum_{k=0}^{s-1} \begin{pmatrix} s \\ k \end{pmatrix} \partial_x^{k+1} E \partial_x^{s-k} (1+m)^{-1/2} \right]^2, \end{aligned}$$

where $\begin{pmatrix} s \\ k \end{pmatrix}$ is a binomial coefficient.

Denoting

$$\lambda := \left(\int_{\mathbb{R}^2} \frac{\left[\partial_x^{s+1} E \right]^2}{1+m} dxdz \right)^{1/2},$$

$$\mu := \left(\int_{\mathbb{R}^2} \left[\sum_{k=0}^{s-1} \begin{pmatrix} s \\ k \end{pmatrix} \partial_x^{k+1} E \partial_x^{s-k} (1+m)^{-1/2} \right]^2 dxdz \right)^{1/2},$$

by the Cauchy-Schwarz inequality, we estimate

$$\begin{aligned} \lambda^2 &\leq J - 2 \int_{\mathbb{R}^2} \frac{\partial_x^{s+1} E}{(1+m)^{1/2}} \sum_{k=0}^{s-1} \begin{pmatrix} s \\ k \end{pmatrix} \partial_x^{k+1} E \partial_x^{s-k} (1+m)^{-1/2} dxdz \\ &\quad - \int_{\mathbb{R}^2} \left[\sum_{k=0}^{s-1} \begin{pmatrix} s \\ k \end{pmatrix} \partial_x^{k+1} E \partial_x^{s-k} (1+m)^{-1/2} \right]^2 dxdz \\ &\leq J + 2\lambda\mu. \end{aligned}$$

But then

$$\lambda^2 - 2\mu\lambda - J \le 0 \quad \Rightarrow \quad \lambda \le \mu + \sqrt{\mu^2 + J}.$$

Let us show that $\mu < \infty$ for any $t \in [0, t_0]$.

By the triangle inequality for L^2-norm, for some constant $C > 0$, we have

$$\begin{aligned}
\mu &\le C \sum_{k=0}^{s-1} \left(\int_{\mathbb{R}^2} \left(\partial_x^{k+1} E \right)^2 \left[\partial_x^{s-k} (1+m)^{-1/2} \right]^2 dxdz \right)^{1/2} \\
&\le C \left[\|\partial_x E\|_{L^\infty} \left\| \partial_x^s (1+m)^{-1/2} \right\|_{L^2} + \sum_{k=1}^{s-1} \left\| \partial_x^{k+1} E \right\|_{L^2} \left\| \partial_x^{s-k} (1+m)^{-1/2} \right\|_{L^\infty} \right].
\end{aligned}$$

The right-hand side of the last inequality is bounded for any $t \in [0, t_0]$ due to finiteness of $\|\partial_x^s E\|_{L^2}$, $\left\|\partial_x^{s+1} m\right\|_{L^2}$, $\|\partial_x E\|_{L^\infty}$, $\left\|\partial_x^{s-1} m\right\|_{L^\infty}$, $\|m\|_{L^\infty}$ by (3.7)-(3.8), the Proposition 2.2 and the Banach algebra property of L^∞-norm.

From here, boundedness of λ follows for all $t \in [0, t_0]$ and yields

$$\left(\int_{\mathbb{R}^2} \frac{\left[\partial_x^{s+1} E\right]^2}{1+m} dxdz \right)^{1/2} < \infty.$$

Now, we notice since $m|_{t=0} = 0$ and $\partial_t m = E^2 \ge 0$, we have $m(x, z, t) \ge 0$ for all $(x, z) \in \mathbb{R}^2$ and $t \in [0, t_0]$. Therefore, we have

$$\frac{1}{1 + \|m\|_{L^\infty}} \int_{\mathbb{R}^2} \left(\left[\partial_x^{s+1} E\right]^2 + \left[\partial_z^{s+1} E\right]^2 \right) dxdz \le \int_{\mathbb{R}^2} \left(\frac{\left[\partial_x^{s+1} E\right]^2 + \left[\partial_z^{s+1} E\right]^2}{1+m} \right) dxdz < \infty,$$

and thus conclude that, for all $t \in [0, t_0]$,

$$\int_{\mathbb{R}^2} \left(\left[\partial_x^{s+1} E\right]^2 + \left[\partial_z^{s+1} E\right]^2 \right) dxdz < \infty. \tag{3.9}$$

It is also clear that the norm in (3.9) is a continuous function of t on $[0, t_0]$ so that the assertion (3.5) holds.

To obtain (3.6), we use the bootstrapping argument for the second equation in the system

(3.1) because the space $H^s\left(\mathbb{R}^2\right)$ forms a Banach algebra for $s > 1$ by the Proposition 2.1. $\square$

The following characterization is useful to extend the local solution of the Lemma 3.1 in the sense that if a local solution exists, it persists on a larger time interval as long as a certain condition is fulfilled. This result is similar to the blow-up criteria of solutions in other equations [2, 11, 15].

Theorem 3.1. *Local solution of the system (3.1) in the Lemma 3.1 does not blow up as $t \to t_0$ if*

$$\sup_{t\in[0,t_0]} \left(\|E\|_{L^\infty} + \|\partial_t E\|_{L^\infty} + \|\nabla E\|_{L^\infty}\right) < \infty. \tag{3.10}$$

Proof. In order to verify the condition (3.10), we suppose

$$M_1 := \sup_{t\in[0,t_0]} \|E\|_{L^\infty} < \infty, \qquad M_2 := \sup_{t\in[0,t_0]} \|\nabla E\|_{L^\infty} < \infty, \qquad M_3 := \sup_{t\in[0,t_0]} \|\partial_t E\|_{L^\infty} < \infty,$$

and show that, for all $t \in [0, t_0]$,

$$\|E\|_{H^4}, \; \|\partial_t E\|_{H^3}, \; \left\|\partial_t^2 E\right\|_{H^2} < \infty.$$

To demonstrate this, we employ the energy method. For the sake of compactness, let us use short notation $E_x := \partial_x E$, $E_t := \partial_t E$, and so on for other derivatives of E and m.

Let us multiply the first equation of the system (3.1) by E_t and integrate by parts employing decay of $E_t E_x$ and $E_t E_z$ to zero as $|x|, |z| \to \infty$ that is justified for the local solution of the Lemma 3.1 for $s = 3$ by the Proposition 2.2. Thus, we obtain

$$\frac{d\mathcal{H}_1}{dt} = \frac{1}{2}\int\limits_{\mathbb{R}^2} E^2 E_t^2 \, dxdz \quad \Rightarrow \quad \frac{d\mathcal{H}_1}{dt} \leq M_1^2 \mathcal{H}_1, \tag{3.11}$$

where we have used the second equation of the system (3.1) and introduced the first energy functional

$$\mathcal{H}_1 := \frac{1}{2}\int\limits_{\mathbb{R}^2} \left((1+m) E_t^2 + E_x^2 + E_z^2\right) dxdz. \tag{3.12}$$

By the Gronwall's inequality (2.9) and the fact that $m(x,z,t) \geq 0$ for all $(x,z) \in \mathbb{R}^2$, we obtain

$$\|E_x\|_{L^2}^2 + \|E_z\|_{L^2}^2 + \|E_t\|_{L^2}^2 \leq 2\mathcal{H}_1 \leq 2\mathcal{H}_1\Big|_{t=0} e^{M_1^2 t} < \infty, \quad t \in [0, t_0].$$

From here, by the Lemma 2.1 for $p = 2$, we also control $\|E\|_{L^2}$ as follows

$$\frac{d}{dt}\|E\|_{L^2} \leq \|E_t\|_{L^2} \quad \Rightarrow \quad \|E\|_{L^2} \leq t_0 \sup_{t\in[0,t_0]} \|E_t\|_{L^2} + \left(\|E\|_{L^2}\right)\big|_{t=0} < \infty, \quad t \in [0, t_0],$$

and thus conclude that $E \in H^1(\mathbb{R}^2)$ and $E_t \in L^2(\mathbb{R}^2)$ for all $t \in [0, t_0]$.

Now, we perform the same procedure but differentiating the first equation of the system (3.1) with respect to x, multiplying it by E_{xt} and integrating over $(x,z) \in \mathbb{R}^2$. Repeating the same with z- and t-variables, we sum the results to obtain

$$\frac{d\mathcal{H}_2}{dt} = \frac{1}{2}\int_{\mathbb{R}^2} \left(E^2\left[E_{xt}^2 + E_{zt}^2 - E_{tt}^2\right] - E_{tt}\left[E_{xt}m_x + E_{zt}m_z\right]\right) dxdz, \tag{3.13}$$

where the second energy functional was introduced

$$\mathcal{H}_2 := \frac{1}{2}\int_{\mathbb{R}^2} \left((1+m)E_{tt}^2 + (2+m)\left[E_{xt}^2 + E_{zt}^2\right] + E_{xx}^2 + E_{zz}^2 + 2E_{xz}^2\right) dxdz, \tag{3.14}$$

and we used the decay of $E_{xt}E_{xx}$, $E_{xt}E_{xz}$, $E_{zt}E_{zz}$, $E_{zt}E_{xz}$, $E_{tt}E_{xt}$ and $E_{tt}E_{zt}$ to zero as $|x|, |z| \to \infty$, which is justified for the local solution of the Lemma 3.1 for $s = 3$ by the Proposition 2.2.

We have

$$\|E_{xx}\|_{L^2}^2 + \|E_{zz}\|_{L^2}^2 + 2\|E_{xz}\|_{L^2}^2 + 2\|E_{xt}\|_{L^2}^2 + 2\|E_{zt}\|_{L^2}^2 + \|E_{tt}\|_{L^2}^2 \leq 2\mathcal{H}_2.$$

We shall now control $\mathcal{H}_2$ from the equation (3.13). The terms in (3.13) with $E^2E_{xt}^2$, $E^2E_{zt}^2$ and $E^2E_{tt}^2$ are controlled by a multiple of $M_1^2\mathcal{H}_2$.

Additionally, we need to bound $\|m\|_{L^\infty}$ and $\|\nabla m\|_{L^\infty}$. By the Corollary 2.1 for $p = \infty$, we have

$$\|m\|_{L^\infty} \leq t_0 \sup_{t\in[0,t_0]} \|m_t\|_{L^\infty} \leq t_0 M_1^2 \tag{3.15}$$

and

$$\|\nabla m\|_{L^\infty} \leq t_0 \sup_{t\in[0,t_0]} \|\nabla m_t\|_{L^\infty} \leq 2t_0 M_1 M_2, \quad t \in [0, t_0], \tag{3.16}$$

where we have used the initial condition $m|_{t=0} = 0$ and the second equation of the system (3.1).

By the triangle inequality and the Cauchy-Schwarz inequality, we have

$$\frac{d\mathcal{H}_2}{dt} \leq M_1 \left(M_1 + 2t_0 M_2\right) \mathcal{H}_2.$$

By the Gronwall's inequality (2.9), we conclude that

$$\mathcal{H}_2 \leq \mathcal{H}_2\bigg|_{t=0} e^{M_1(M_1+2t_0M_2)t} < \infty, \quad t \in [0, t_0].$$

Thus, we deduce that $E \in H^2\left(\mathbb{R}^2\right)$, $E_t \in H^1\left(\mathbb{R}^2\right)$ and $E_{tt} \in L^2\left(\mathbb{R}^2\right)$ for all $t \in [0, t_0]$.

Now, we continue in the same manner as before, operating on with the first equation of the system (3.1) by $E_{xxt}\partial_x^2 + E_{zzt}\partial_z^2 + E_{ttt}\partial_t^2$ and integrating in (x, z) over $\mathbb{R}^2$ by parts to reduce the expression to first-order derivatives of m only. In the end, we obtain we obtain a functional that is not positive definite. Its boundedness does not yield a bound on the norms of derivatives of E it includes. To remedy the situation, we add $\int\limits_{\mathbb{R}^2} \left[m_x^2 + m_z^2\right] E_{tt}^2 dxdz$ to the energy functional and hence obtain

$$\begin{aligned} \frac{d\mathcal{H}_3}{dt} &= \frac{1}{2}\int\limits_{\mathbb{R}^2}\bigg(E^2\left[E_{xxt}^2 + E_{zzt}^2 - 3E_{ttt}^2\right] - 2m_x\left[E_{xxt}E_{xtt} + E_{xxx}E_{ttt}\right] \\ &- 2m_z\left[E_{zzt}E_{ztt} + E_{zzz}E_{ttt}\right] - 4E_{tt}E\left[E_{ttt}E_t + E_{xxx}E_x + E_{zzz}E_z\right] \\ &+ 8EE_{tt}^2\left[E_x m_x + E_z m_z\right] + 4E_{tt}E_{ttt}\left[m_x^2 + m_z^2\right]\bigg)dxdz, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned}\mathcal{H}_3 &= \frac{1}{2}\int_{\mathbb{R}^2}\Big((1+m)\left[E_{ttt}^2+E_{xxt}^2+E_{zzt}^2\right]+E_{xtt}^2+E_{ztt}^2+\frac{1}{2}\left(E_{xxx}^2+E_{zzz}^2\right)\\&+ E_{xxz}^2+E_{xzz}^2+\frac{1}{2}\left[E_{xxx}-2m_xE_{tt}\right]^2+\frac{1}{2}\left[E_{zzz}-2m_zE_{tt}\right]^2\Big)dxdz. \qquad (3.18)\end{aligned}$$

We have

$$\begin{aligned}&\|E_{xxx}\|_{L^2}^2+\|E_{zzz}\|_{L^2}^2+2\|E_{xxz}\|_{L^2}^2+2\|E_{xzz}\|_{L^2}^2+2\|E_{xxt}\|_{L^2}^2+2\|E_{zzt}\|_{L^2}^2\\&\qquad+2\|E_{ttt}\|_{L^2}^2+2\|E_{xtt}\|_{L^2}^2+2\|E_{ztt}\|_{L^2}^2 \le 4\mathcal{H}_3.\end{aligned}$$

In deriving the balance equation (3.17), we have used the decay of $E_{xxx}E_{xxt}$, $E_{xxz}E_{xxt}$, $E_{zzz}E_{zzt}$, $E_{xzz}E_{zzt}$, $E_{ttt}E_{xtt}$, $E_{ttt}E_{ztt}$, $m_xE_{xxt}E_{tt}$ and $m_zE_{zzt}E_{tt}$ to zero as $|x|\to\infty$, $|z|\to\infty$. This decay can be obtained by working with approximation sequences as follows.

Let us consider an approximation of the initial conditions E_0, E_1 by the sequences of functions $\left\{E_0^{(n)}\right\}_{n=1}^{\infty}\in H^5\left(\mathbb{R}^2\right)$, $\left\{E_1^{(n)}\right\}_{n=1}^{\infty}\in H^4\left(\mathbb{R}^2\right)$, respectively. Then, by the Lemma 3.1 for $s=4$, the corresponding sequence of local solutions will be

$$E^{(n)}\in C\left([0,t_0],H^5\left(\mathbb{R}^2\right)\right)\cap C^1\left([0,t_0],H^4\left(\mathbb{R}^2\right)\right)\cap C^2\left([0,t_0],H^3\left(\mathbb{R}^2\right)\right).$$

The decay assumptions are valid by the Proposition 2.2 for the approximate solution $E^{(n)}$.

Because the space $H^5\left(\mathbb{R}^2\right)$ is dense in $H^4\left(\mathbb{R}^2\right)$ and so is $H^4\left(\mathbb{R}^2\right)$ in $H^3\left(\mathbb{R}^2\right)$, we have

$$\left\|E_0^{(n)}-E_0\right\|_{H^4}\to 0,\quad \left\|E_1^{(n)}-E_1\right\|_{H^3}\to 0\quad \text{as } n\to\infty,$$

and hence, by the continuous dependence of the solution on the initial data in the Lemma 3.1, we have

$$\left\|E^{(n)}-E\right\|_{H^4}\to 0,\quad \left\|E_t^{(n)}-E_t\right\|_{H^3}\to 0\quad \text{as } n\to\infty,$$

that holds for all $t\in[0,t_0]$.

This approximation argument furnishes the required decay of solutions at infinity in the justification of the energy balance (3.17).

Using (3.16), we estimate the m-dependent terms in (3.17) as follows

$$\begin{aligned}\int_{\mathbb{R}^2} m_x \left[E_{xxt}E_{xtt} + E_{xxx}E_{ttt}\right] dxdz &\leq \|\nabla m\|_{L^\infty} \left(\|E_{xxt}\|_{L^2}\|E_{xtt}\|_{L^2} + \|E_{xxx}\|_{L^2}\|E_{ttt}\|_{L^2}\right) \\ &\leq 8t_0 M_1 M_2 \mathcal{H}_3,\end{aligned}$$

$$\int_{\mathbb{R}^2} E_{tt}E_{ttt}m_x^2 dxdz \leq \|\nabla m\|_{L^\infty}^2 \|E_{tt}\|_{L^2}\|E_{ttt}\|_{L^2} \leq 8t_0^2 M_1^2 M_2^2 \mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2},$$

$$\int_{\mathbb{R}^2} EE_xE_{tt}^2 m_x dxdz \leq M_1M_2 \|\nabla m\|_{L^\infty} \|E_{tt}\|_{L^2}^2 \leq 4t_0 M_1^2 M_2^2 \mathcal{H}_2,$$

and similarly for the z-derivatives terms.

The estimates of the m-independent terms in (3.17) are straightforward

$$\int_{\mathbb{R}^2} E^2 \left[E_{xxt}^2 + E_{zzt}^2 - 3E_{ttt}^2\right] dxdz \leq M_1^2 \mathcal{H}_3,$$

$$\int_{\mathbb{R}^2} EE_{tt}\left[E_{xxx}E_x + E_{zzz}E_z + E_{ttt}E_t\right] dxdz \leq M_1\left(2M_2 + M_3\right)\mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2}.$$

Therefore,

$$\frac{d\mathcal{H}_3}{dt} \leq M_1\left(M_1 + 16t_0M_2\right)\mathcal{H}_3 + 2M_1\left(2M_2 + M_3 + 16t_0^2M_1M_2^2\right)\mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2} + 32t_0M_1^2M_2^2\mathcal{H}_2.$$

By the inequality $\mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2} \leq \frac{1}{2}\left(\mathcal{H}_2 + \mathcal{H}_3\right)$, we have

$$\frac{d\mathcal{H}_3}{dt} \leq F\mathcal{H}_3 + G\mathcal{H}_2,$$

where

$$\begin{aligned} F &:= M_1\left(M_1 + 18t_0M_2 + t_0M_3 + 16t_0^3M_1M_2^2\right), \\ G &:= M_1\left(2M_2 + M_3 + 16t_0^2M_1M_2^2\right) + 32t_0M_1^2M_2^2. \end{aligned}$$

Then, by the Gronwall's inequality (2.9), for $t \in [0, t_0]$, we bound

$$\mathcal{H}_3 \le \left(\mathcal{H}_3\Big|_{t=0} + Gt_0 \sup_{t\in[0,t_0]} \mathcal{H}_2\right) e^{tF} < \infty.$$

Thus, we deduce that $E \in H^3(\mathbb{R}^2)$, $E_t \in H^2(\mathbb{R}^2)$ and $E_{tt} \in H^1(\mathbb{R}^2)$ for all $t \in [0, t_0]$.

We proceed to obtain the final energy estimates. We act by

$$E_{xxxt}\partial_x^3 + E_{zzzt}\partial_z^3 + E_{xttt}\partial_x\partial_t^2 + E_{zttt}\partial_z\partial_t^2 + E_{tttt}\partial_t^3$$

on the first equation of the system (3.1) and integrate the result in (x, z) over $\mathbb{R}^2$. Following the same steps as in the previous energy level computations, we can introduce the positive definite energy functional

$$\begin{aligned} \mathcal{H}_4 &:= \frac{1}{2}\int_{\mathbb{R}^2}\Big[(1+m)\left(E_{xxxt}^2 + E_{zzzt}^2 + E_{xttt}^2 + E_{zttt}^2 + E_{tttt}^2\right) + \frac{1}{2}\left(E_{xxxx}^2 + E_{zzzz}^2\right) \\ &+ 2E_{xxzz}^2 + 2E_{xztt}^2 + E_{xxtt}^2 + E_{zztt}^2 + E_{xttt}^2 + E_{zttt}^2 + \frac{1}{2}\left(E_{xxxx} - 2m_{xx}E_{tt}\right)^2 \\ &+ \frac{1}{2}\left(E_{zzzz} - 2m_{zz}E_{tt}\right)^2\Big]\, dxdz \end{aligned} \tag{3.19}$$

to obtain

$$\begin{aligned} \frac{d\mathcal{H}_4}{dt} &\le \frac{1}{2}\int_{\mathbb{R}^2}\left[E^2\left(E_{xxxt}^2 + E_{zzzt}^2 - 3E_{xttt}^2 - 3E_{zttt}^2 - 4E_{tttt}^2\right) - 4E_{tt}\left(E_x^2E_{xxxx} + E_z^2E_{zzzz} + E_t^2E_{tttt}\right)\right. \\ &- 4EE_{tt}\left(E_{xx}E_{xxxx} + E_{zz}E_{zzzz} + E_{tt}E_{tttt}\right) - 4m_{xx}\left(E_{xtt}E_{xxxt} + E_{ttt}E_{xxxx}\right) \\ &- 4m_{zz}\left(E_{ztt}E_{zzzt} + E_{ttt}E_{zzzz}\right) - 6m_xE_{xxtt}E_{xxxt} - 6m_zE_{zztt}E_{zzzt} - 12EE_tE_{ttt}E_{tttt} \\ &+ \left. 4E_{ttt}E_{tt}\left(m_{xx}^2 + m_{zz}^2\right) + 8E_{tt}^2\left(m_{xx}\left(E_x^2 + EE_{xx}\right) + m_{zz}\left(E_z^2 + EE_{zz}\right)\right)\right] dxdz. \end{aligned} \tag{3.20}$$

This computation is valid under assumption on decay to zero of $E_{xxxt}E_{xxxx}$, $E_{xxxt}E_{xxxz}$, $E_{zzzt}E_{zzzz}$, $E_{zzzt}E_{xzzz}$, $E_{tttt}E_{xttt}$, $E_{tttt}E_{zttt}$, $E_{xttt}E_{xxtt}$, $E_{xttt}E_{xztt}$, $E_{zttt}E_{zztt}$, $E_{zttt}E_{xztt}$, $m_{xx}E_{xxtt}E_{tt}$ and $m_{zz}E_{zztt}E_{tt}$ as $|x| \to \infty$, $|z| \to \infty$, but this decay can be justified by the approximation argument for a sequence of local solutions of the Lemma 3.1 for $s = 5$ as done in the previous energy level computations.

We have

$$\begin{aligned}\|E_{xxxx}\|_{L^2}^2 + \|E_{zzzz}\|_{L^2}^2 + 4\|E_{xxzz}\|_{L^2}^2 + 2\|E_{xxxt}\|_{L^2}^2 + 2\|E_{zzzt}\|_{L^2}^2 \\ +2\|E_{xxtt}\|_{L^2}^2 + 2\|E_{zztt}\|_{L^2}^2 + 4\|E_{xztt}\|_{L^2}^2 \quad \leq \quad 4\mathcal{H}_4. \end{aligned} \tag{3.21}$$

We use the Proposition 2.2 to bound

$$\|E_{xx}\|_{L^\infty} \leq C_0 \left(\|\Delta E_{xx}\|_{L^2} + \|E_{xx}\|_{L^2}\right) \leq \sqrt{2}C_0 \left(\mathcal{H}_4^{1/2} + \mathcal{H}_2^{1/2}\right),$$

for some $C_0 > 0$.

Using this estimate and the Corollary 2.1, from the second equation of the system (3.1), we obtain, for $t \in [0, t_0]$,

$$\|m_{xx}\|_{L^\infty} \leq t_0 \sup_{[0,t]} \|m_{txx}\|_{L^\infty} \leq 2t_0 M_2^2 + 2\sqrt{2}C_0 t_0 M_1 \left(\sup_{[0,t]} \mathcal{H}_4^{1/2} + \sup_{t\in[0,t_0]} \mathcal{H}_2^{1/2}\right).$$

The similar estimates hold for the z-derivatives terms. Lengthy and tedious calculations result in

$$\frac{d\mathcal{H}_4}{dt} \leq I\mathcal{H}_4 + J \sup_{[0,t]} \mathcal{H}_4 + L, \tag{3.22}$$

where I, J, L are some coefficients that depend on t_0, M_1, M_2, M_3, $\sup_{t\in[0,t_0]} \mathcal{H}_2$ and $\sup_{t\in[0,t_0]} \mathcal{H}_3$.

The last inequality can be rewritten in integral form as

$$\sup_{[0,t]} \mathcal{H}_4 \leq \left.\mathcal{H}_4\right|_{t=0} + t_0 L + (I + J) \int_0^t \sup_{[0,\tau]} \mathcal{H}_4 d\tau.$$

By the Gronwall's inequality, we bound

$$\sup_{t\in[0,t_0]} \mathcal{H}_4 \le [\mathcal{H}_4|_{t=0} + t_0 L]\, e^{t(I+J)} < \infty.$$

Now, since

$$\|E_{xxxz}\|_{L^2}^2, \; \|E_{xzzz}\|_{L^2}^2 \le \frac{1}{2}\|E_{xxxx}\|_{L^2}^2 + \frac{1}{2}\|E_{xxzz}\|_{L^2}^2,$$

$$\|E_{xxzt}\|_{L^2}^2, \; \|E_{xzzt}\|_{L^2}^2 \le \frac{1}{2}\|E_{xxxt}\|_{L^2}^2 + \frac{1}{2}\|E_{zzzt}\|_{L^2}^2,$$

which is a result of straightforward estimates on the Fourier transform side, from (3.21), we conclude that $E \in H^4(\mathbb{R}^2)$, $E_t \in H^3(\mathbb{R}^2)$ and $E_{tt} \in H^2(\mathbb{R}^2)$ for all $t \in [0, t_0]$. □

Remark 3.1. To eliminate finite-time blow-up of the component m in H^4-norm, we can use the estimate of E and the Proposition 2.1 applied to the second equation of the system (3.1).

3.2 Local well-posedness of the NLS system

Let us now consider the question of the well-posedness of the system (1.5)-(1.6). For the sake of brevity, we will work with the rescaled equations

$$\begin{cases} \partial_X^2 A + i(\partial_T A + \partial_Z A) + m_0 A = 0, \\ \partial_T m_0 = |A|^2, \end{cases} \qquad (X,Z)\in\mathbb{R}^2,\ T\in\mathbb{R}_+, \tag{3.23}$$

subject to the initial data $A|_{T=0} = A_0 \in H^s(\mathbb{R}^2)$, $m_0|_{T=0} = 0$, for some integer $s \ge 2$.

Theorem 3.2. *For any integer $s \ge 2$ and $\delta > 2\sup_{T\in\mathbb{R}_+} \|A_0\|_{H^s(\mathbb{R}^2)}$, there exist a positive constant T_0 and a unique solution $A \in C([0,T_0], H^s(\mathbb{R}^2)) \cap C^1([0,T_0], H^{s-2}(\mathbb{R}^2))$ to the system (3.23) such that $A|_{T=0} = A_0$ and $\sup_{T\in[0,T_0]} \|A\|_{H^s(\mathbb{R}^2)} \le \delta$.*

Proof. Let us take Fourier transform in both spatial variables, and denote

$$\widehat{m_0 A}(\xi,\eta,T) := \frac{1}{2\pi}\int_{\mathbb{R}^2} m_0(X,Z,T)\, A(X,Z,T)\, e^{i(\xi X + \eta Z)} dX dZ.$$

The first equation then becomes

$$\partial_T \hat{A} = i\left(-\xi^2 + \eta\right)\hat{A} + i\widehat{m_0 A},$$

which leads to

$$\hat{A}(\xi,\eta,T) = \hat{A}_0(\xi,\eta)\, e^{i\left(-\xi^2+\eta\right)T} + i\int_0^T e^{i\left(-\xi^2+\eta\right)(T-\tau)}\widehat{m_0 A}(\xi,\eta,\tau)\, d\tau. \tag{3.24}$$

Introduce the Schrödinger kernel

$$S_T(X) := \frac{1}{\sqrt{4\pi T}} e^{-\frac{i\pi}{4}} e^{\frac{iX^2}{4T}}.$$

Then, since

$$\mathcal{F}^{-1}\left[e^{i\left(-\xi^2+\eta\right)T}\right] = S_T(X)\,\delta(T-Z),$$

the inverse Fourier transform of (3.24) results in the integral equation

$$A(X,Z,T) = S_T(X)\star A_0(X,Z-T) + i\int_0^T S_{T-\tau}(X)\star\left[m_0(X,Z-T+\tau,\tau)\, A(X,Z-T+\tau,\tau)\right] d\tau,$$

where $\star$ stands for convolution in X-variable.

Making use of the second equation in (3.23) and $m_0|_{T=0} = 0$, we can rewrite the last line in the form of operator equation

$$A(X,Z,T) = K\left[A(X,Z,T)\right], \tag{3.25}$$

where

$$K[A] := S_T(X)\star A_0(X,Z-T) + i\int_0^T S_{T-\tau}(X)\star\left[A(X,Z-T+\tau,\tau)\int_0^\tau \left|A(X,Z-T+\tilde{\tau},\tilde{\tau})\right|^2 d\tilde{\tau}\right] d\tau.$$

We will obtain local well-posedness of the system (3.23), once we are able to apply the Proposition 2.5 to conclude the existence and uniqueness of solutions to the integral equation

(3.25).

Let us show that the condition of the Banach fixed-point theorem is fulfilled in a closed ball of radius δ in the space $C\left([0,T_0], H^s\left(\mathbb{R}^2\right)\right)$ for some $s \geq 2$, $\delta > 0$ and $T_0 > 0$:

$$\bar{\mathcal{B}}_\delta := \left\{ f \in C\left([0,T_0], H^s\left(\mathbb{R}^2\right)\right) : \sup_{T\in[0,T_0]} \|f\|_{H^s(\mathbb{R}^2)} \leq \delta \right\}. \tag{3.26}$$

First of all, we need to show that map $\bar{\mathcal{B}}_\delta$ is an invariant subspace of the operator K, that is, for any $A \in \bar{\mathcal{B}}_\delta \subset C\left([0,T_0], H^s\left(\mathbb{R}^2\right)\right)$

$$\sup_{T\in[0,T_0]} \|K[A]\|_{H^s(\mathbb{R}^2)} \leq \delta \tag{3.27}$$

holds for suitable choice of $\delta > 0$ and $T_0 > 0$. Then, we need to show that K is a contractive operator in the sense that there is $q \in (0,1)$ such that for any $A^{(1)}$, $A^{(2)} \in \bar{\mathcal{B}}_\delta$

$$\sup_{T\in[0,T_0]} \left\| K\left[A^{(1)}\right] - K\left[A^{(2)}\right] \right\|_{H^s(\mathbb{R}^2)} \leq q \sup_{T\in[0,T_0]} \left\| A^{(1)} - A^{(2)} \right\|_{H^s(\mathbb{R}^2)}. \tag{3.28}$$

To choose $\delta > 0$ and $T_0 > 0$ such that both conditions (3.27)-(3.28) are satisfied, we proceed with analysis on the Fourier transform side using (3.24) rather than (3.25).

We start by showing (3.27). Let $A \in \bar{\mathcal{B}}_\delta$, that is, $\sup_{T\in[0,T_0]} \|A\|_{H^s(\mathbb{R}^2)} \leq \delta$. Then, applying the Plancherel's theorem and the Minkowski integral inequality to (3.24), we obtain

$$\begin{aligned}
\sup_{T\in[0,T_0]} \|K[A]\|_{H^s(\mathbb{R}^2)} &= \sup_{T\in[0,T_0]} \left\| \left(1+\xi^2+\eta^2\right)^{s/2} \hat{K}[A] \right\|_{L^2(\mathbb{R}^2)} \\
&\leq \left\| \left(1+\xi^2+\eta^2\right)^{s/2} \hat{A}_0 \right\|_{L^2(\mathbb{R}^2)} + \sup_{T\in[0,T_0]} \int_0^T \left\| \left(1+\xi^2+\eta^2\right)^{s/2} \widehat{m_0 A} \right\|_{L^2(\mathbb{R}^2)} d\tau \\
&= \|A_0\|_{H^s(\mathbb{R}^2)} + \sup_{T\in[0,T_0]} \int_0^T \|m_0 A\|_{H^s(\mathbb{R}^2)} d\tau.
\end{aligned}$$

Employing the Proposition 2.1 and the Corollary 2.2, we arrive at

$$\begin{aligned}\sup_{T\in[0,T_0]}\|K[A]\|_{H^s(\mathbb{R}^2)} &\leq \|A_0\|_{H^s(\mathbb{R}^2)}+C_s^2T_0^2\sup_{T\in[0,T_0]}\|A\|^3_{H^s(\mathbb{R}^2)}\\ &\leq \|A_0\|_{H^s(\mathbb{R}^2)}+C_s^2T_0^2\delta^3,\end{aligned}$$

for some constant $C_s>0$. Choosing $\delta\geq 2\|A_0\|_{H^s(\mathbb{R}^2)}$ and $T_0\leq\frac{1}{\sqrt{2}C_s\delta}$, the both terms become less or equal to $\delta/2$ which furnishes (3.27).

Now, we proceed with showing (3.28). We write

$$m_0\left(A^{(1)}\right)=m_0\left(A^{(2)}\right)-\left(m_0\left(A^{(2)}\right)-m_0\left(A^{(1)}\right)\right)$$

and using the triangle inequality, the Banach algebra property of $H^s\left(\mathbb{R}^2\right)$ and the same arguments as above, we obtain

$$\begin{aligned}\sup_{T\in[0,T_0]}\left\|K\left[A^{(1)}\right]-K\left[A^{(2)}\right]\right\|_{H^s(\mathbb{R}^2)} &\leq \sup_{T\in[0,T_0]}\int_0^T\left\|m_0\left(A^{(1)}\right)A^{(1)}-m_0\left(A^{(2)}\right)A^{(2)}\right\|_{H^s}d\tau\\ &\leq C_s^2T_0^2\sup_{T\in[0,T_0]}\left[\left\|A^{(2)}\right\|_{H^s}\left\|A^{(1)}-A^{(2)}\right\|_{H^s}\right.\\ &+\left.\left\|A^{(1)}\right\|_{H^s}\left(\left\|A^{(1)}\right\|_{H^s}+\left\|A^{(2)}\right\|_{H^s}\right)\left\|A^{(1)}-A^{(2)}\right\|_{H^s}\right]\\ &\leq 3C_s^2T_0^2\delta^2\sup_{T\in[0,T_0]}\left\|A^{(1)}-A^{(2)}\right\|_{H^s(\mathbb{R}^2)}.\end{aligned}$$

From here, the contraction requirement results in restriction $T_0<\frac{1}{\sqrt{3}C_s\delta}$.

Combining this with the previous condition, we conclude that the choice

$$\delta>2\|A_0\|_{H^s(\mathbb{R}^2)},\quad T_0\leq\frac{1}{\sqrt{3}C_s\delta}$$

leads to the existence of unique solution A of the equation (3.23) in the ball (3.26).

Then, expressing $\partial_T A$ from the first equation of the system, the bootstrapping argument gives $A\in C^1\left([0,T_0],H^{s-2}\left(\mathbb{R}^2\right)\right)$. □

Remark 3.2. Tracing the proof, it is straightforward to see that the same result holds in the presence of inhomogeneous terms in the first equation of (3.23) providing these terms belong to the space $C\left([0,T_0],H^s\left(\mathbb{R}^2\right)\right)$.

Chapter 4

Rigorous justification analysis

4.1 Near-identity transformations

Smallness of remainder terms $U(x,z,t)$ and $N(x,z,t)$ in the decompositions (1.8)-(1.9) hinges on smallness of the right-hand side terms in the system of residual equations (1.10)-(1.11). The right-hand side terms can be made smaller by perfoming appropriate near-identity transformation.

Let us start with the source term $\epsilon^6 \left(R_6^{(U)}\right)_{\omega_0}$ in the first equation (1.10) and introduce

$$U_1(x,z,t) := U(x,z,t) - \epsilon^4 \left(F(X,Z,T)\right)_{\omega_0}, \tag{4.1}$$

where $F(X,Z,T)$ will be chosen later.

Eliminating $U(x,z,t)$ from (1.10), we obtain

$$\partial_x^2 U_1 + \partial_z^2 U_1 - \left(1 + \epsilon^2 m_0 + N\right) \partial_t^2 U_1 = -\epsilon^2 \left(\tilde{R}_2^{(U)}\right)_{\omega_0} N - \epsilon^6 \left(\tilde{R}_6^{(U)}\right)_{\omega_0} - \epsilon^8 \left(\tilde{R}_8^{(U)}\right)_{\omega_0},$$

where

$$\begin{aligned}
\tilde{R}_2^{(U)} &:= R_2^{(U)} + \epsilon^2 \left(\omega_0^2 F + 2i\omega_0 \epsilon^2 \partial_T F - \epsilon^4 \partial_T^2 F\right), \\
\tilde{R}_6^{(U)} &:= \partial_X^2 F + 2i\omega_0 \left(\partial_Z F + \partial_T F\right) + \omega_0^2 m_0 F - \left(\partial_T^2 A - \partial_Z^2 A - 2i\omega_0 m_0 \partial_T A\right), \\
\tilde{R}_8^{(U)} &:= \partial_Z^2 F - \partial_T^2 F - m_0 \partial_T^2 A + 2i\omega_0 m_0 \partial_T F - \epsilon^2 m_0 \partial_T^2 F.
\end{aligned}$$

From here one can see that the $\mathcal{O}\left(\epsilon^6\right)$ source term can be eliminated (i.e. $\tilde{R}_6^{(U)} = 0$)

providing that $F(X,Z,T)$ solves the linear inhomogeneous Schrödinger equation

$$\partial_X^2 F + 2i\omega_0 \left(\partial_Z F + \partial_T F\right) + \omega_0^2 m_0 F = \partial_T^2 A - \partial_Z^2 A - 2i\omega_0 m_0 \partial_T A.$$

Hence, the equation for $U_1(x,z,t)$ has a $\mathcal{O}\left(\epsilon^8\right)$ source term. Generally, such transformation can be repeated k times to have a source term of order $\mathcal{O}\left(\epsilon^{6+2k}\right)$.

Now, we proceed with the second equation (1.11) treating it separately. To remove the $\mathcal{O}\left(\epsilon^4\right)$ source term, we introduce

$$N_1 := N - \epsilon^4 \left(\frac{A^2}{2i\omega_0}\right)_{2\omega_0} \tag{4.2}$$

and obtain the equation with the $\mathcal{O}\left(\epsilon^6\right)$ source term

$$\partial_t N_1 = -\epsilon^6 \left(\frac{A\partial_T A}{i\omega_0}\right)_{2\omega_0} + 2\epsilon^2 (A)_{\omega_0} U + U^2.$$

In a similar fashion, this transformation can be repeated n times to get the $\mathcal{O}\left(\epsilon^{4+2n}\right)$ source term.

We can also improve the second term in (1.11) by performing another type of the near-identity transformation

$$N_2 := N - 2\epsilon^2 \left(\frac{A}{i\omega_0}\right)_{\omega_0} U, \tag{4.3}$$

in which case we obtain

$$\partial_t N_2 = \epsilon^4 \left(A^2\right)_{2\omega_0} - 2\epsilon^2 \left(\frac{A}{i\omega_0}\right)_{\omega_0} \partial_t U - 2\epsilon^4 \left(\frac{\partial_T A}{i\omega_0}\right)_{\omega_0} U + U^2.$$

Such transformation move the linear term in U to the $\mathcal{O}\left(\epsilon^4\right)$ order, whereas the $\mathcal{O}\left(\epsilon^2\right)$ term depends now on $\partial_t U$ which norm is expected to be smaller. Note that repetition of this transformation generally is not effective because smallness of a norm of $\partial_t^2 U$ in comparison with norm of $\partial_t U$ is not anticipated.

The last two near-identity transformations (4.2)-(4.3) can be combined in a straightforward way, however putting together near-identity transformations (4.1), (4.2) and (4.3) should be done more carefully due to intertwining structure of the equations.

For justification analysis, we will need the double transformation (4.2) for the source term in the N equation, the transformation (4.3) of the linear term, and the single transformation (4.1) for the source term in the U equation. Due to dependence on N, the transformation in the U equation should be modified by including third-harmonic term which in turn changes the equation for N. The resulting near-identity transformation is implemented by introducing

$$\begin{aligned} V &:= U - \epsilon^4 (B)_{\omega_0} - \epsilon^4 (D)_{3\omega_0}, && (4.4)\\ M &:= N - \epsilon^4 N_0 + 2\epsilon^2 \left(\frac{A}{i\omega_0}\right)_{\omega_0} V + \epsilon^4 \left(\frac{A^2}{2i\omega_0}\right)_{2\omega_0} + \epsilon^6 R_6^{(M)}, && (4.5) \end{aligned}$$

where

$$R_6^{(M)} := \frac{2\left(A\bar{B} - \bar{A}B\right)}{i\omega_0} - \left(\frac{A\partial_T A}{2\omega_0^2} - \frac{AB + \bar{A}D}{i\omega_0}\right)_{2\omega_0} + \left(\frac{AD}{2i\omega_0}\right)_{4\omega_0}.$$

Here, bar denotes complex conjugation, and $B(X,Z,T)$, $D(X,Z,T)$, $N_0(X,Z,T)$ solve the following linear inhomogeneous equations

$$\begin{aligned} \partial_X^2 B + 2i\omega_0 \left(\partial_Z B + \partial_T B\right) + \omega_0^2 m_0 B &= \partial_T^2 A - \partial_Z^2 A - 2i\omega_0 m_0 \partial_T A - \frac{i\omega_0}{2} A^2 \bar{A}, && (4.6)\\ \partial_X^2 D + 2i\omega_0 \left(\partial_Z D + \partial_T D\right) + \omega_0^2 m_0 D &= -\frac{i\omega_0}{2} A^3, && (4.7)\\ \partial_T N_0 &= 2\left(A\bar{B} + \bar{A}B\right). && (4.8) \end{aligned}$$

As a result of these transformations, the system of residual equations (1.10)-(1.11) transforms to the system

$$\partial_x^2 V + \partial_z^2 V - \left(1 + \epsilon^2 m_0 + N\right) \partial_t^2 V = -\epsilon^2 \omega_0^2 (A)_{\omega_0} M + 2i\omega_0 \epsilon^4 |A|^2 V - \epsilon^8 R_8^{(V)}, \quad (4.9)$$

$$\partial_t M = \epsilon^8 R_8^{(M)} + 2\epsilon^4 R_4^{(M)} V + 2\epsilon^2 \left(\frac{A}{i\omega_0}\right)_{\omega_0} \partial_t V + V^2, \quad (4.10)$$

where

$$R_4^{(M)} := \left(\frac{\partial_T A}{i\omega_0} + B\right)_{\omega_0} + (D)_{3\omega_0},$$

$$R_8^{(M)} := -\frac{\partial_T\left(\bar{A}B - A\bar{B}\right)}{2i\omega_0} + \left(B\bar{D} - \frac{\partial_T\left(A\partial_T A\right)}{2\omega_0^2} + \frac{\partial_T\left(AB + \bar{A}D\right)}{i\omega_0}\right)_{2\omega_0} + \left(BC + \frac{\partial_T\left(AD\right)}{2i\omega_0}\right)_{4\omega_0},$$

$$\begin{aligned} R_8^{(V)} \;:=\; & \left(\frac{i\omega_0}{2}\left[2\left|A\right|^2 B + A^2\bar{B} + 7\bar{A}D + 4m_0\partial_T B\right] + \frac{\left|A\right|^2\partial_T A}{2} + A^2\partial_T\bar{A} - m_0\partial_T^2 A + \partial_Z^2 B - \partial_T^2 B\right)_{\omega_0} \\ - & \left(\frac{i\omega_0}{2}\left[\left|A\right|^2 D + A^2 B - 12m_0\partial_T D\right] + \frac{A^2\partial_T A}{2} - \partial_Z^2 D + \partial_T^2 D\right)_{3\omega_0} + 3\left(i\omega_0 A^2 D\right)_{5\omega_0}. \end{aligned}$$

This system of equations is a starting point in our justification analysis.

4.2 Local control of the residual terms

We now proceed with the estimates of the residual terms $U(x,z,t)$, $N(x,z,t)$ in the decompositions (1.8)-(1.9) given sufficiently smooth initial data. The amplitudes A and m_0 change on the temporal scale of $T = \epsilon^2 t$ on $[0, T_0]$. Therefore, the validity of approximation needs to be justified for all $t \in \left[0, T_0/\epsilon^2\right]$.

We would like to prove that there are α_0, $\beta_0 > 0$ such that

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|U(\cdot,\cdot,t)\right\|_{L^2} = \mathcal{O}\left(\epsilon^{\beta_0}\right), \tag{4.11}$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|N(\cdot,\cdot,t)\right\|_{L^2} = \mathcal{O}\left(\epsilon^{\alpha_0}\right). \tag{4.12}$$

At the same time, the leading order approximation in the decompositions (1.8)-(1.9) is estimated to be

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|\epsilon^2 (A)_{\omega_0}\right\|_{L^2} \le 2\epsilon^2 \sup_{T\in[0,T_0]} \left(\int_{\mathbb{R}^2} \left|A\left(\epsilon x, \epsilon^2 z, T\right)\right|^2 dxdz\right)^{1/2} = \mathcal{O}\left(\epsilon^{1/2}\right),$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|\epsilon^2 m_0\right\|_{L^2} = \epsilon^2 \sup_{T\in[0,T_0]} \left(\int_{\mathbb{R}^2} \left|m_0\left(\epsilon x, \epsilon^2 z, T\right)\right|^2 dxdz\right)^{1/2} = \mathcal{O}\left(\epsilon^{1/2}\right).$$

Therefore, the error terms in the decompositions (1.8)-(1.9) are smaller than the leading order terms in L^2-norm if $\alpha_0,\ \beta_0 > \frac{1}{2}$.

Note that we are loosing $\epsilon^{3/2}$ in norm of $\epsilon^2 (A)_{\omega_0}$ due to integration because A depends on the slow variables $X = \epsilon x$, $Z = \epsilon^2 z$. The same is true when computing L^2-norms of the other terms $R_8^{(V)}$, $R_8^{(M)}$, $R_4^{(M)}$ which absolute value depend only on slow variables X, Z.

As before, we will use index notation for partial derivatives and let C denote a generic positive constant. Also, we will employ subscript notation such as $\|\cdot\|_{L^2_{X,Z}}$, $\nabla_{(X,Z)}$ and $\Delta_{(X,Z)}$ when necessary to emphasize that a norm or derivatives are computed with respect to slow variables.

Using the near-identity transformations (4.4) and (4.5), under assumptions B, D, $N_0 \in L^2\left(\mathbb{R}^2\right)$, $A \in L^\infty\left(\mathbb{R}^2\right) \cap L^2\left(\mathbb{R}^2\right)$, $R_6^{(M)} \in L^2\left(\mathbb{R}^2\right)$, we can see that

$$\sup_{t\in[0,T_0/\epsilon^2]} \|U(\cdot,\cdot,t)\|_{L^2} \leq \sup_{t\in[0,T_0/\epsilon^2]} \|V(\cdot,\cdot,t)\|_{L^2} + C\epsilon^{5/2},$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|N(\cdot,\cdot,t)\|_{L^2} \leq \sup_{t\in[0,T_0/\epsilon^2]} \|M(\cdot,\cdot,t)\|_{L^2} + C\epsilon^2 \sup_{t\in[0,T_0/\epsilon^2]} \|V(\cdot,\cdot,t)\|_{L^2} + C\epsilon^{5/2}.$$

Hence, to have (4.11)-(4.12) with $\alpha_0,\ \beta_0 > \frac{1}{2}$, we need

$$\sup_{t\in[0,T_0/\epsilon^2]} \|V(\cdot,\cdot,t)\|_{L^2} = \mathcal{O}\left(\epsilon^\beta\right), \tag{4.13}$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|M(\cdot,\cdot,t)\|_{L^2} = \mathcal{O}\left(\epsilon^\alpha\right), \tag{4.14}$$

with

$$\alpha,\ \beta > \frac{1}{2}. \tag{4.15}$$

4.2.1 First energy level

While $M(x,z,t)$ can be controlled directly from the equation (4.10), the estimate of $V(x,z,t)$ relies on the energy approach used in the Section 3.1. Multiplication of the equation (4.9) by $V_t(x,z,t)$ and further integration by parts in (x,z) over $\mathbb{R}^2$ lead to

$$\frac{d\mathcal{H}_1}{dt} = \int_{\mathbb{R}^2} \left(\epsilon^4 |A|^2 V_t^2 + \frac{1}{2} N_t V_t^2 + \epsilon^2 \omega_0^2 (A)_{\omega_0} M V_t - 2i\omega_0 \epsilon^4 |A|^2 V V_t + \epsilon^8 R_8^{(V)} V_t \right) dxdz, \tag{4.16}$$

where we introduced the first energy functional

$$\mathcal{H}_1 := \frac{1}{2} \int_{\mathbb{R}^2} \left[\left(1 + \epsilon^2 m_0 + N\right) V_t^2 + V_x^2 + V_z^2 \right] dxdz. \tag{4.17}$$

This yields the estimate

$$\begin{aligned} \frac{d\mathcal{H}_1}{dt} &\leq 2\epsilon^4 \|A\|_{L^\infty}^2 \mathcal{H}_1 + \|N_t\|_{L^\infty} \mathcal{H}_1 + 2\sqrt{2}\epsilon^2 \omega_0^2 \|A\|_{L^\infty} \|M\|_{L^2} \mathcal{H}_1^{1/2} \\ &+ 2\sqrt{2}\epsilon^4 \omega_0 \|A\|_{L^\infty}^2 \|V\|_{L^2} \mathcal{H}_1^{1/2} + \sqrt{2}\epsilon^{13/2} \left\| R_8^{(V)} \right\|_{L^2_{X,Z}} \mathcal{H}_1^{1/2}. \end{aligned} \tag{4.18}$$

Let $Q_1 := \mathcal{H}_1^{1/2}$ and assume that we can prove

$$\sup_{t \in [0, T_0/\epsilon^2]} Q_1 = \mathcal{O}\left(\epsilon^{\delta_1}\right), \tag{4.19}$$

for some $\delta_1 > 0$.

Then, since $V|_{t=0}$, the Corollary 2.1 implies, for $t \in \left[0, T_0/\epsilon^2\right]$,

$$\|V\|_{L^2} \leq \frac{T_0}{\epsilon^2} \sup_{t \in [0, T_0/\epsilon^2]} \|V_t\|_{L^2} \leq \frac{\sqrt{2}T_0}{\epsilon^2} \sup_{t \in [0, T_0/\epsilon^2]} Q_1,$$

and hence $\sup_{t \in [0, T/\epsilon^2]} \|V\|_{L^2} = \mathcal{O}\left(\epsilon^{\delta_1 - 2}\right)$, that is, $\beta = \delta_1 - 2$ in (4.13).

Similarly, by the Proposition 2.3 with $\sigma = 1$, we estimate the nonlinear term in (4.10)

$$\|V\|_{L^4}^2 \leq C_\sigma \|V\|_{L^2} \|\nabla V\|_{L^2}.$$

Since $M|_{t=0} = 0$, the Corollary 2.1 implies

$$\begin{aligned}
\|M\|_{L^2} &\leq \frac{T_0}{\epsilon^2} \sup_{t\in[0,T_0/\epsilon^2]} \|M_t\|_{L^2} \leq \epsilon^{9/2} T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_8^{(M)} \right\|_{L^2_{X,Z}} \\
&+ 2\sqrt{2} T_0 \left(T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_4^{(M)} \right\|_{L^\infty} + \frac{2}{\omega_0} \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \right) \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \\
&+ 2C_\sigma T_0^2 \epsilon^{-4} \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right)^2 ,
\end{aligned}$$

and thus $\sup_{t\in[0,T/\epsilon^2]} \|M\|_{L^2} = \mathcal{O}\left(\epsilon^{9/2} + \epsilon^{\delta_1} + \epsilon^{2\delta_1 - 4}\right)$, that is, in (4.14),

$$\alpha = \min\left\{ \frac{9}{2},\, \delta_1,\, 2\delta_1 - 4 \right\}. \tag{4.20}$$

To control the energy, we need to bound $\|N_t\|_{L^\infty}$. This can be done under additional assumptions that will be verified after.

First of all, from (1.11), we estimate

$$\|N_t\|_{L^\infty} \leq 2\epsilon^4 \|A\|_{L^\infty}^2 + 4\epsilon^2 \|A\|_{L^\infty} \|U\|_{L^\infty} + \|U\|_{L^\infty}^2 , \tag{4.21}$$

where $\|U\|_{L^\infty}$ is controlled using (4.4)

$$\|U\|_{L^\infty} \leq 2\epsilon^4 \|B\|_{L^\infty} + 2\epsilon^4 \|D\|_{L^\infty} + \|V\|_{L^\infty} . \tag{4.22}$$

By the Proposition 2.2, we can bound

$$\|V\|_{L^\infty} \leq C \|V\|_{H^2} , \tag{4.23}$$

if we assume the L^2-norm of second derivatives of V is controlled by some quantity Q_2 to be introduced later, that is

$$\|V_{xx}\|_{L^2}, \|V_{zz}\|_{L^2} \leq \sqrt{2} Q_2, \qquad \sup_{t\in[0,T_0/\epsilon^2]} Q_2 = \mathcal{O}\left(\epsilon^{\delta_2}\right), \tag{4.24}$$

for some $\delta_2 > 0$, then, we have, for some $C_0 > 0$,

$$\|V\|_{L^\infty} \leq C_0 \left[Q_2 + T_0 \epsilon^{-2} \sup_{t \in [0, T_0/\epsilon^2]} Q_1 \right]. \tag{4.25}$$

As we will see, $\sup_{t\in[0,T/\epsilon^2]} \|V\|_{L^\infty}$ is always bigger than $\mathcal{O}\left(\epsilon^4\right)$, so we certainly have $\|U\|_{L^\infty} \leq \|V\|_{L^\infty}$. Using the last estimate, from (4.21), we can thus obtain

$$\begin{aligned} \|N_t\|_{L^\infty} &\leq 2\epsilon^4 \|A\|_{L^\infty}^2 + 4C_0 \epsilon^2 \|A\|_{L^\infty} Q_2 + C_0^2 Q_2^2 \\ &+ \epsilon^{-4} C_0^2 T_0^2 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right)^2 + 2C_0 T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \left(\epsilon^{-2} C_0 Q_2 + 2 \|A\|_{L^\infty} \right). \end{aligned} \tag{4.26}$$

These bounds applied to (4.18) yield

$$\frac{dQ_1}{dt} \leq I_1 Q_1 + J_1,$$

where

$$\begin{aligned} I_1 &:= 4\epsilon^4 \left(\sup_{T\in[0,T_0]} \|A\|_{L^\infty} \right)^2 + 2\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2 + \frac{C_0^2}{2} \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2 \right)^2 \\ &+ \epsilon^{-2} C_0^2 T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_2 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 + \frac{C_0^2 T_0^2}{2} \epsilon^{-4} \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right)^2 \\ &+ 2C_0 T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_1, \end{aligned}$$

$$\begin{aligned} J_1 &:= \sqrt{2}\epsilon^{13/2} \left(\frac{1}{2} \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_8^{(V)} \right\|_{L^2_{X,Z}} + \omega_0^2 T_0 \sup_{t\in[0,T_0/\epsilon^2]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_8^{(M)} \right\|_{L^2_{X,Z}} \right) \\ &+ 2\epsilon^2 \omega_0 T_0 \left(\left(4+\sqrt{2}\right) \left(\sup_{T\in[0,T_0]} \|A\|_{L^\infty} \right)^2 + 2T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_4^{(M)} \right\|_{L^\infty} \right) \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \\ &+ 2\sqrt{2}\epsilon^{-2} \omega_0^2 T_0^2 C_\sigma \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right)^2. \end{aligned}$$

By the Proposition 2.4, we have, for $t \in \left[0, T_0/\epsilon^2\right]$,

$$Q_1 \leq T_0 \epsilon^{-2} J_1 e^{I_1 T_0 \epsilon^{-2}}. \tag{4.27}$$

To prevent divergence of the exponential factor $e^{I_1 T_0 \epsilon^{-2}}$ as $\epsilon \to 0$, we impose the condition

$$\min\left\{\delta_2,\, 2\delta_2 - 2,\, \delta_2 + \delta_1 - 4,\, 2\delta_1 - 6,\, \delta_1 - 2\right\} \geq 0. \tag{4.28}$$

Also, we require $\delta_1 > 4$ so the quadratic term $\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1\right)^2$ in $T_0\epsilon^{-2}J_1$ is negligibly small. On the other hand, the $\mathcal{O}\left(\epsilon^{13/2}\right)$ free term in J_1 restricts smallness of $\sup_{t\in[0,T_0/\epsilon^2]} Q_1$. Indeed, by (4.27), we have

$$\begin{aligned}
\sup_{t\in[0,T_0/\epsilon^2]} Q_1 \;\; \leq \;\; & e^{I_1 T_0 \epsilon^{-2}} \left[\sqrt{2}\epsilon^{9/2} T_0 \left(\frac{1}{2} \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_8^{(V)} \right\|_{L^2_{X,Z}} \right.\right. \\
& + \left. \omega_0^2 T_0 \sup_{t\in[0,T_0/\epsilon^2]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_8^{(M)} \right\|_{L^2_{X,Z}} \right) \\
& + \left. 2\omega_0 T_0^2 \left(\left(\sqrt{2} + 4\right) \left(\sup_{t\in[0,T_0/\epsilon^2]} \|A\|_{L^\infty} \right)^2 + 2T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_4^{(M)} \right\|_{L^\infty} \right) \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right],
\end{aligned}$$

which bounds $\sup_{t\in[0,T_0/\epsilon^2]} Q_1 = \mathcal{O}\left(\epsilon^{9/2}\right)$ providing T_0 is small enough such that

$$2\omega_0 T_0^2 e^{I_1 T_0 \epsilon^{-2}} \left(\left(\sqrt{2} + 4\right) \left(\sup_{t\in[0,T_0/\epsilon^2]} \|A\|_{L^\infty} \right)^2 + 2T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_4^{(M)} \right\|_{L^\infty} \right) < 1.$$

Therefore, the conditions (4.20), (4.28) imply that

$$\alpha = \delta_1 = \frac{9}{2}, \quad \beta = \delta_1 - 2 = \frac{5}{2}, \tag{4.29}$$

and we additionally require

$$\delta_2 \geq 1. \tag{4.30}$$

We will ensure that this constraint on δ_2 is satisfied by continuing next with estimates on the second derivatives of V.

4.2.2 Second energy level

Acting on the equation (4.9) with operator $V_{xt}\partial_x + V_{zt}\partial_z + V_{tt}\partial_t$ and integrating in (x,z) over $\mathbb{R}^2$, we introduce the second energy functional that shall be controlled

$$\mathcal{H}_2 := \frac{1}{2}\int_{\mathbb{R}^2} \left(\left(1+\epsilon^2 m_0 + N\right) V_{tt}^2 + \left(2+\epsilon^2 m_0 + N\right)\left[V_{xt}^2 + V_{zt}^2\right] + V_{xx}^2 + V_{zz}^2 + 2V_{xz}^2\right) dxdz. \tag{4.31}$$

Long but straightforward computations show that the rate of change of the second energy functional is given by

$$\frac{d\mathcal{H}_2}{dt} = \sum_{n=1}^{9} K_n, \tag{4.32}$$

where

$$K_1 := \epsilon^4 \int_{\mathbb{R}^2} |A|^2 \left(V_{tt}^2 + V_{tx}^2 + V_{tz}^2\right) dxdz,$$
$$K_2 := -\epsilon^3 \int_{\mathbb{R}^2} V_{tt} \left(V_{tx}\partial_X m_0 + \epsilon V_{tz}\partial_Z m_0 + 2\epsilon |A|^2 V_{tt}\right) dxdz,$$
$$K_3 := \frac{1}{2}\int_{\mathbb{R}^2} N_t \left(V_{tx}^2 + V_{tz}^2 - V_{tt}^2\right) dxdz,$$
$$K_4 := \int_{\mathbb{R}^2} N \left[V_{ttx}V_{tx} + V_{ttz}V_{tz} + V_{tt}\left(V_{txx} + V_{tzz}\right)\right] dxdz,$$
$$K_5 := \epsilon^2\omega_0^2 \int_{\mathbb{R}^2} (A)_{\omega_0} \left(M_x V_{tx} + M_z V_{tz} + M_t V_{tt}\right) dxdz,$$
$$K_6 := \epsilon^2\omega_0^2 \int_{\mathbb{R}^2} M\left[\epsilon \left(A_X\right)_{\omega_0} V_{tx} + \left(\left(i\omega_0 A\right)_{\omega_0} + \epsilon^2 \left(A_Z\right)_{\omega_0}\right) V_{tz} + \left(-\left(i\omega_0 A\right)_{\omega_0} + \epsilon^2 \left(A_T\right)_{\omega_0}\right) V_{tt}\right] dxdz,$$
$$K_7 := -2i\epsilon^5\omega_0 \int_{\mathbb{R}^2} V \left(V_{tx}\partial_X |A|^2 + \epsilon V_{tz}\partial_Z |A|^2 + \epsilon V_{tt}\partial_T |A|^2\right) dxdz,$$
$$K_8 := -2i\epsilon^4\omega_0 \int_{\mathbb{R}^2} |A|^2 \left(V_{tx}V_x + V_{tz}V_z + V_{tt}V_t\right) dxdz,$$
$$K_9 := \epsilon^8 \int_{\mathbb{R}^2} \left(\epsilon V_{tx}\partial_X R_8^{(V)} + V_{tz}\partial_z R_8^{(V)} + V_{tt}\partial_t R_8^{(V)}\right) dxdz.$$

Let $Q_2 := \mathcal{H}_2^{1/2}$, and we want to ensure that

$$\sup_{t\in[T_0/\epsilon^2]} Q_2 = \mathcal{O}\left(\epsilon^{\delta_2}\right), \tag{4.33}$$

for some $\delta_2 \geq 1$ according to (4.30).

We estimate the terms in (4.32) as follows

$$\begin{aligned}
|K_1| &\leq 2\epsilon^4 \|A\|_{L^\infty}^2 \mathcal{H}_2,\\
|K_2| &\leq 4\epsilon^3 \left(\left\|\nabla_{(X,Z)} m_0\right\|_{L^\infty} + \epsilon \|A\|_{L^\infty}^2\right)\mathcal{H}_2,\\
|K_3| &\leq \|N_t\|_{L^\infty} \mathcal{H}_2,\\
|K_4| &\leq \|N\|_{L^\infty} \left(\|V_{ttx}\|_{L^2} + \|V_{ttz}\|_{L^2} + \|V_{xxt}\|_{L^2} + \|V_{zzt}\|_{L^2}\right)\mathcal{H}_2^{1/2},\\
|K_5| &\leq 2\sqrt{2}\epsilon^2\omega_0^2 \|A\|_{L^\infty} \left(2\|\nabla M\|_{L^2} + \|M_t\|_{L^2}\right)\mathcal{H}_2^{1/2},\\
|K_6| &\leq 2\sqrt{2}\epsilon^2\omega_0^2 \|M\|_{L^2} \left(\epsilon\|A_X\|_{L^\infty} + \epsilon^2\|A_Z\|_{L^\infty} + \epsilon^2\|A_T\|_{L^\infty} + 2\omega_0\|A\|_{L^\infty}\right)\mathcal{H}_2^{1/2},\\
|K_7| &\leq 4\epsilon^5\omega_0 \|A\|_{L^\infty}\|V\|_{L^2} \left(\|A_X\|_{L^\infty} + \epsilon\|A_Z\|_{L^\infty} + \epsilon\|A_T\|_{L^\infty}\right)\mathcal{H}_2^{1/2},\\
|K_8| &\leq 12\epsilon^4\omega_0 \|A\|_{L^\infty}^2 \mathcal{H}_1^{1/2}\mathcal{H}_2^{1/2},\\
|K_9| &\leq \sqrt{2}\epsilon^{13/2}\left(\epsilon\left\|\partial_X R_8^{(V)}\right\|_{L^2_{X,Z}} + \left\|\partial_z R_8^{(V)}\right\|_{L^2_{X,Z}} + \left\|\partial_t R_8^{(V)}\right\|_{L^2_{X,Z}}\right)\mathcal{H}_2^{1/2}.
\end{aligned}$$

To proceed further, we shall use the bounds

$$\begin{aligned}
\sup_{T\in[0,T_0]} \left\|\nabla_{(X,Z)} m_0\right\|_{L^\infty} &\leq T_0 \sup_{T\in[0,T_0]} \left\|\nabla_{(X,Z)}\partial_T m_0\right\|_{L^\infty}\\
&\leq 4T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{T\in[0,T_0]} \left\|\nabla_{(X,Z)} A\right\|_{L^\infty},
\end{aligned}$$

$$\begin{aligned}
\sup_{t\in[0,T_0/\epsilon^2]} \|\nabla M\|_{L^2} &\leq 2\sqrt{2}T_0 \left[T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\|\nabla R_4^{(M)}\right\|_{L^\infty} + C_0 T_0 \epsilon^{-4} \sup_{t\in[0,T_0/\epsilon^2]} Q_1\right.\\
&+ \left. C_0\epsilon^{-2} \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right] \sup_{t\in[0,T_0/\epsilon^2]} Q_1\\
&+ \frac{4\sqrt{2}T_0}{\omega_0} \sup_{T\in[0,T_0]} \left\|\nabla_{(X,Z)} A\right\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2,
\end{aligned} \tag{4.34}$$

where we dropped terms which are of higher order of smallness under assumptions $\nabla R_8^{(M)} \in L^2(\mathbb{R}^2)$, $R_4^{(M)} \in L^\infty(\mathbb{R}^2)$, $\nabla A \in L^\infty(\mathbb{R}^2)$.

In order to control $\|N\|_{L^\infty}$, we can use (4.26) and the Proposition 2.1

$$\|N\|_{L^\infty} \leq \epsilon^{-2} T_0 \sup_{t\in[0,T_0/\epsilon^2]} \|N_t\|_{L^\infty}.$$

Additionally, to estimate the K_4 term, we need to bound the third derivatives V_{ttx}, V_{ttz}, V_{xxt}, V_{zzt}, which L^2-norms are controlled in terms of the energy $\mathcal{H}_3$ that will be introduced further, that is

$$\|V_{ttx}\|_{L^2}, \|V_{ttz}\|_{L^2}, \|V_{xxt}\|_{L^2}, \|V_{zzt}\|_{L^2} \leq \sqrt{2}\mathcal{H}_3^{1/2}, \quad \sup_{t\in[0,T_0/\epsilon^2]} \mathcal{H}_3^{1/2} = \mathcal{O}\left(\epsilon^{\delta_3}\right), \tag{4.35}$$

for some $\delta_3 > 0$.

Then,

$$\begin{aligned} |K_4| \quad \leq \quad & 2T_0 \left[2\epsilon^2 \left(\sup_{T\in[0,T_0]} \|A\|_{L^\infty} \right)^2 + 4C_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2 \right. \\ + \quad & \epsilon^{-2} C_0^2 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2 \right)^2 + \epsilon^{-6} C_0 T_0^2 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right)^2 + 2C_0^2 T_0 \epsilon^{-4} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2 \\ + \quad & \left. 4C_0 T_0 \epsilon^{-2} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \right] \mathcal{H}_3^{1/2} Q_2. \end{aligned}$$

Details for other K-terms in (4.32) can be elaborated using the bounds above. Neglecting source terms in K_9 in comparison with other terms, we obtain

$$\frac{dQ_2}{dt} \leq I_2 Q_2 + J_2, \tag{4.36}$$

where

$$I_2 \quad := \quad 4\epsilon^3 T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{T\in[0,T_0]} \|\nabla A\|_{L^\infty} + 2\epsilon^2 C_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2$$

$$
\begin{aligned}
&+ \quad \frac{C_0^2}{2}\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2 + \frac{\epsilon^{-4}C_0^2T_0^2}{2}\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1\right)^2 + 2\epsilon^{-2}C_0T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \\
&+ \quad \epsilon^{-4}C_0^2T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2,
\end{aligned}
$$

$$
\begin{aligned}
J_2 \quad := \quad & T_0 \sup_{t\in[0,T_0/\epsilon^2]} \mathcal{H}_3^{1/2}\Bigg[2\epsilon^2\left(\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\right)^2 + 4C_0 \sup_{T\in[0,T_0]}\|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2 \\
+ \quad & \epsilon^{-2}C_0^2\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2 + \epsilon^{-6}C_0T_0^2\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1\right)^2 + 2\epsilon^{-4}C_0^2T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2 \\
+ \quad & 4\epsilon^{-2}C_0T_0 \sup_{T\in[0,T_0]}\|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_1\Bigg] \\
+ \quad & 4\omega_0^2T_0 \sup_{T\in[0,T_0]}\|A\|_{L^\infty}\Bigg[2\sup_{T\in[0,T_0]}\|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \\
+ \quad & \sqrt{2}C_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2 + \epsilon^{-2}\frac{T_0}{\sqrt{2}}\left(2C_0 + C_\sigma\omega_0\right)\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1\right)^2 \\
+ \quad & \frac{2}{\omega_0}\epsilon^2 \sup_{T\in[0,T_0]}\|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2\Bigg].
\end{aligned}
$$

The Proposition 2.4 applied to (4.36) gives

$$Q_2 \leq T_0\epsilon^{-2}J_2e^{I_2T_0\epsilon^{-2}}.$$

To bound the exponential factor, we require $I_2\epsilon^{-2}$ to be finite, that is

$$\min\{\delta_2,\, 2\delta_2 - 2,\, 2\delta_1 - 6,\, \delta_1 + \delta_2 - 6,\, \delta_1 - 4\} \geq 0, \tag{4.37}$$

which is further reduced to the condition

$$\delta_2 \geq 6 - \delta_1. \tag{4.38}$$

Then, we obtain $\sup_{t\in[0,T_0/\epsilon^2]} Q_2 = \mathcal{O}\left(\epsilon^{5/2}\right)$, that is,

$$\delta_2 = \frac{5}{2}. \tag{4.39}$$

To bound terms in $J_2\epsilon^{-2}$, we require

$$\min\{\delta_1+\delta_2-2,\, 2\delta_1-4,\, \delta_1-2\} \geq \delta_2, \tag{4.40}$$

$$\min\{\delta_3,\, \delta_2+\delta_3-2,\, 2\delta_1+\delta_3-8,\, \delta_1+\delta_2+\delta_3-6,\, \delta_1+\delta_3-4\} \geq \delta_2, \tag{4.41}$$

$$2\delta_2+\delta_3-4 > \delta_2, \tag{4.42}$$

and T_0 to be sufficiently small such that

$$4\omega_0 T_0^2 e^{I_2 T_0 \epsilon^{-2}} \left[2\left(\sup_{T\in[0,T_0]} \|A\|_{L^\infty} \right)^2 + \sqrt{2}\epsilon^{-2} C_0\omega_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right] < 1.$$

Taking into account (4.29) and (4.40)-(4.42), we find a constraint on δ_3

$$\delta_3 \geq \delta_2. \tag{4.43}$$

Now, we proceed on the next energy level to verify (4.43) and finalize justification estimates.

4.2.3 Third energy level

To get the next energy level estimates, we consider the operator $V_{xxt}\partial_x^2 + V_{zzt}\partial_z^2 + V_{ttt}\partial_t^2$ acting on (4.9), which is followed by integration in (x,z) over $\mathbb{R}^2$.

Following the same steps as before, we introduce the third energy functional

$$\mathcal{H}_3 := \frac{1}{2}\int_{\mathbb{R}^2} \Big(\left(1+\epsilon^2 m_0 + N\right)\left(V_{xxt}^2 + V_{zzt}^2 + V_{ttt}^2\right) + \frac{1}{2}\left(V_{xxx}^2 + V_{zzz}^2\right) + V_{xxz}^2 + V_{xzz}^2$$

$$+ \quad \frac{1}{2}\left[V_{xxx} - 2N_x V_{tt}\right]^2 + \frac{1}{2}\left[V_{zzz} - 2N_z V_{tt}\right]^2\Big) \, dxdz \tag{4.44}$$

such that (4.9) yields

$$\frac{d\mathcal{H}_3}{dt} = \sum_{n=1}^{15} L_n, \tag{4.45}$$

where

$$\begin{aligned}
L_1 &:= \epsilon^4 \int_{\mathbb{R}^2} |A|^2 \left(V_{xxt}^2 + V_{zzt}^2 - V_{ttt}^2\right) dxdz,\\
L_2 &:= \frac{1}{2} \int_{\mathbb{R}^2} N_t \left(V_{xxt}^2 + V_{zzt}^2 - 3V_{ttt}^2\right) dxdz,\\
L_3 &:= -\epsilon^4 \int_{\mathbb{R}^2} V_{tt} \left(V_{xxt}\partial_X^2 m_0 + \epsilon^2 V_{zzt}\partial_Z^2 m_0 + \epsilon^2 V_{ttt}\partial_T |A|^2\right) dxdz,\\
L_4 &:= -2\epsilon^3 \int_{\mathbb{R}^2} \left(V_{xxt}V_{xtt}\partial_X m_0 + \epsilon V_{zzt}V_{ztt}\partial_Z m_0\right) dxdz,\\
L_5 &:= -\int_{\mathbb{R}^2} \left[N_x \left(2V_{xxt}V_{xtt} + V_{xxx}V_{ttt} - V_{xtt}^2\right) + N_z \left(2V_{zzt}V_{ztt} + V_{zzz}V_{ttt} - V_{ztt}^2\right)\right] dxdz,\\
L_6 &:= \int_{\mathbb{R}^2} \left[V_{tt}\left(N_{xt}V_{xxx} + N_{zt}V_{zzz}\right) + V_{tt}^2 \left(N_{tt} + 2N_xN_{xt} + 2N_zN_{zt}\right)\right] dxdz,\\
L_7 &:= 2\int_{\mathbb{R}^2} V_{tt}V_{ttt}\left(N_x^2 + N_z^2\right) dxdz,\\
L_8 &:= \epsilon^2\omega_0^2 \int_{\mathbb{R}^2} (A)_{\omega_0} \left(M_{xx}V_{xxt} + M_{zz}V_{zzt} + M_{tt}V_{ttt}\right) dxdz,\\
L_9 &:= 2\epsilon^3 \int_{\mathbb{R}^2} \left(i\omega_0^3 A\right)_{\omega_0} \left(M_z V_{zzz} - M_t V_{ttt}\right) dxdz,\\
L_{10} &:= 2\epsilon^2\omega_0^2 \int_{\mathbb{R}^2} \left((A_X)_{\omega_0} M_x V_{xxt} + \epsilon (A_Z)_{\omega_0} M_z V_{zzt} + \epsilon (A_T)_{\omega_0} M_t V_{ttt}\right) dxdz,\\
L_{11} &:= \epsilon^4\omega_0^2 \int_{\mathbb{R}^2} M \left((A_{XX})_{\omega_0} V_{xxt} + \epsilon^2 (A_{ZZ})_{\omega_0} V_{zzt} + \epsilon^2 (A_{TT})_{\omega_0} V_{ttt}\right) dxdz,\\
L_{12} &:= -2i\epsilon^6\omega_0 \int_{\mathbb{R}^2} V \left(V_{xxt}\partial_X^2 |A|^2 + \epsilon^2 V_{zzt}\partial_Z^2 |A|^2 + \epsilon^2 V_{ttt}\partial_T^2 |A|^2\right) dxdz,\\
L_{13} &:= -4i\epsilon^5\omega_0 \int_{\mathbb{R}^2} \left(V_x V_{xxt}\partial_X |A|^2 + \epsilon V_z V_{zzt}\partial_Z |A|^2 + \epsilon V_t V_{ttt}\partial_T |A|^2\right) dxdz,\\
L_{14} &:= -2i\epsilon^4\omega_0 \int_{\mathbb{R}^2} |A|^2 \left(V_{xx}V_{xxt} + V_{zz}V_{zzt} + V_{tt}V_{ttt}\right) dxdz,\\
L_{15} &:= \epsilon^8 \int_{\mathbb{R}^2} \left(\epsilon^2 V_{xxt}\partial_X^2 R_8^{(V)} + V_{zzt}\partial_z^2 R_8^{(V)} + V_{ttt}\partial_t^2 R_8^{(V)}\right) dxdz.
\end{aligned}$$

We shall estimate these terms in the following manner

$$|L_1| \leq 2\epsilon^4 \|A\|_{L^\infty}^2 \mathcal{H}_3,$$

$$|L_2| \leq \|N_t\|_{L^\infty} \mathcal{H}_3,$$

$$|L_3| \leq 2\epsilon^4 \left(\left\|\partial_X^2 m_0\right\|_{L^\infty} + \epsilon^2 \left\|\partial_Z^2 m_0\right\|_{L^\infty} + 2\epsilon^2 \|A\|_{L^\infty} \|A_T\|_{L^\infty}\right) \mathcal{H}_3,$$

$$|L_4| \leq 4\epsilon^3 \left(\|\partial_X m_0\|_{L^\infty} + \epsilon \|\partial_Z m_0\|_{L^\infty}\right) \mathcal{H}_3,$$

$$|L_5| \leq 12 \|\nabla N\|_{L^\infty} \mathcal{H}_3,$$

$$|L_6| \leq 4 \|\nabla N_t\|_{L^\infty} \mathcal{H}_2^{1/2} \mathcal{H}_3^{1/2} + 2\left(\|N_{tt}\|_{L^\infty} + 8 \|\nabla N\|_{L^\infty} \|\nabla N_t\|_{L^\infty}\right) \mathcal{H}_2,$$

$$|L_7| \leq 16 \|\nabla N\|_{L^\infty}^2 \mathcal{H}_2^{1/2} \mathcal{H}_3^{1/2},$$

$$|L_8| \leq 2\sqrt{2}\epsilon^2 \omega_0^2 \|A\|_{L^\infty} \left(\|\Delta M\|_{L^2} + \|M_{tt}\|_{L^2}\right) \mathcal{H}_3^{1/2},$$

$$|L_9| \leq 4\sqrt{2}\epsilon^3 \omega_0^3 \|A\|_{L^\infty} \left(\|\nabla M\|_{L^2} + \|M_t\|_{L^2}\right) \mathcal{H}_3^{1/2},$$

$$|L_{10}| \leq 4\sqrt{2}\epsilon^2 \omega_0^2 \left[\left(\|A_X\|_{L^\infty} + \epsilon \|A_Z\|_{L^\infty}\right) \|\nabla M\|_{L^2} + \epsilon \|A_T\|_{L^\infty} \|M_t\|_{L^2}\right] \mathcal{H}_3^{1/2},$$

$$|L_{11}| \leq 2\sqrt{2}\epsilon^4 \omega_0^2 \left(\|A_{XX}\|_{L^\infty} + \epsilon^2 \|A_{ZZ}\|_{L^\infty} + \epsilon^2 \|A_{TT}\|_{L^\infty}\right) \mathcal{H}_3^{1/2},$$

$$|L_{12}| \leq 4\sqrt{2}\epsilon^6 \omega_0 \|V\|_{L^2} \Big(\|A\|_{L^\infty} \left[\|A_{XX}\|_{L^\infty} + \epsilon^2 \|A_{ZZ}\|_{L^\infty} + \epsilon^2 \|A_{TT}\|_{L^\infty}\right]$$
$$+ \|A_X\|_{L^\infty}^2 + \epsilon^2 \|A_Z\|_{L^\infty}^2 + \epsilon^2 \|A_T\|_{L^\infty}^2\Big) \mathcal{H}_3^{1/2},$$

$$|L_{13}| \leq 32\epsilon^5 \omega_0 \|A\|_{L^\infty} \left(\|A_X\|_{L^\infty} + \epsilon \|A_Z\|_{L^\infty} + \epsilon \|A_T\|_{L^\infty}\right) \mathcal{H}_1^{1/2} \mathcal{H}_3^{1/2},$$

$$|L_{14}| \leq 12\epsilon^4 \omega_0 \|A\|_{L^\infty}^2 \mathcal{H}_2^{1/2} \mathcal{H}_3^{1/2},$$

$$|L_{15}| \leq \sqrt{2}\epsilon^{13/2} \left(\epsilon^2 \left\|\partial_X^2 R_8^{(V)}\right\|_{L^2} + \left\|\partial_z^2 R_8^{(V)}\right\|_{L^2} + \left\|\partial_t^2 R_8^{(V)}\right\|_{L^2}\right) \mathcal{H}_3^{1/2}.$$

At this energy level there will be no restriction on upper bound of time-interval, therefore we do not necessarily need to keep track of particular expressions of all the L-terms estimates, instead we will be looking at their order of smallness only.

To control the right-hand side of (4.45), we need (4.34). Also, we use the Corollary 2.1 and the Propositions 2.2 and 2.3 to obtain the following estimates

$$\sup_{t \in [0, T_0/\epsilon^2]} \|\nabla N\|_{L^\infty} \leq 2T_0 \epsilon^{-2} \sup_{t \in [0, T_0/\epsilon^2]} \left[\epsilon^4 \|A\|_{L^\infty}^2 + 2\epsilon^2 \|A\|_{L^\infty} \|U\|_{L^\infty} + \|\nabla U\|_{L^\infty}^2 + \|U\|_{L^\infty} \|\nabla U\|_{L^\infty}\right],$$

$$\begin{aligned}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|\Delta M\|_{L^2} &\leq T_0\epsilon^{-2}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\left[\epsilon^{13/2}\left\|\Delta R_8^{(M)}\right\|_{L^2_{X,Z}}+2\epsilon^4\left\|\Delta R_4^{(M)}\right\|_{L^\infty}\|V\|_{L^2}\right.\\ &+ 2\epsilon^4\left\|R_4^{(M)}\right\|_{L^\infty}\|\Delta V\|_{L^2}+4\epsilon^2\omega_0\|A\|_{L^\infty}\|V_t\|_{L^2}+4\frac{\epsilon^2}{\omega_0}\|A\|_{L^\infty}\|\Delta V_t\|_{L^2}\\ &+\left. 2\|V\|_{L^\infty}\|\Delta V\|_{L^2}+2C_\sigma\|\nabla V\|_{L^2}\|\Delta V\|_{L^2}\right],\end{aligned}$$

$$\begin{aligned}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|M_{tt}\|_{L^2} &\leq \epsilon^{13/2}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\left\|\Delta R_8^{(M)}\right\|_{L^2_{X,Z}}+2\epsilon^4\sup_{t\in\left[0,T_0/\epsilon^2\right]}\left\|R_4^{(M)}\right\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|\partial_t V\|_{L^2}\\ &+ 2\epsilon^4\sup_{t\in\left[0,T_0/\epsilon^2\right]}\left\|\partial_t R_4^{(M)}\right\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|V\|_{L^2}+4\epsilon^2\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|V_t\|_{L^2}\\ &+ 4\frac{\epsilon^2}{\omega_0}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|A\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|V_{tt}\|_{L^2}+2\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|V\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|V_t\|_{L^2},\end{aligned}$$

$$\begin{aligned}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|\nabla N_t\|_{L^\infty} &\leq 4\epsilon^4\omega_0\left(\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\right)^2+4\epsilon^2\omega_0\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|U\|_{L^\infty}\\ &+ 4\epsilon^2\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|\nabla U\|_{L^\infty}+2\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|\nabla U\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|U\|_{L^\infty},\end{aligned}$$

$$\begin{aligned}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|N_{tt}\|_{L^\infty} &\leq 4\epsilon^4\omega_0\left(\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\right)^2+4\epsilon^2\omega_0\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|U\|_{L^\infty}\\ &+ 4\epsilon^2\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|\nabla U\|_{L^\infty}+2\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|\nabla U\|_{L^\infty}\sup_{t\in\left[0,T_0/\epsilon^2\right]}\|U\|_{L^\infty},\end{aligned}$$

$$\sup_{T\in[0,T_0]}\left\|\Delta_{(X,Z)}m_0\right\|_{L^\infty} \leq 4T_0\left[\left(\sup_{T\in[0,T_0]}\|\nabla A\|_{L^\infty}\right)^2+\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\sup_{T\in[0,T_0]}\|\Delta A\|_{L^\infty}\right],$$

where smaller terms are neglected under assumption ΔA, ∇A, $\partial_T A\in L^\infty\left(\mathbb{R}^2\right)$.

Taking into account (4.29), (4.39)-(4.43), we can drop a priori smaller terms and hence obtain

$$\begin{aligned}\frac{d\mathcal{H}_3}{dt} &\leq 24T_0\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\left[\epsilon^2\sup_{T\in[0,T_0]}\|A\|_{L^\infty}+2\sup_{t\in[0,T_0/\epsilon^2]}Q_2\right]\mathcal{H}_3\\ &+ 8\omega_0T_0\sup_{T\in[0,T_0]}\|A\|_{L^\infty}\left[\epsilon^{-2}C_0\omega_0T_0\sup_{t\in[0,T_0/\epsilon^2]}Q_1\sup_{t\in[0,T_0/\epsilon^2]}Q_2\right.\\ &+ \left. C_0\omega_0\left(\sup_{t\in[0,T_0/\epsilon^2]}Q_2\right)^2+4\epsilon^2\sup_{T\in[0,T_0]}\|\nabla A\|_{L^\infty}\sup_{t\in[0,T_0/\epsilon^2]}Q_2\right]\mathcal{H}_3^{1/2}\end{aligned}$$

$$+ \quad 8\epsilon^2\omega_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2 \left(\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty}\right.$$
$$\left. + \quad C_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right). \tag{4.46}$$

We set $Q_3 := \mathcal{H}_3^{1/2}$ and neglect the free term assuming

$$\min\{4+2\delta_2,\, 2+3\delta_2\} > 2\delta_3 + \delta_2. \tag{4.47}$$

Then,

$$\frac{dQ_3}{dt} \quad \leq \quad I_3 Q_3 + J_3, \tag{4.48}$$

where

$$I_3 := 12T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left[\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} + 2 \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right],$$

$$J_3 \quad := \quad 4\omega_0 T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left[\epsilon^{-2} C_0\omega_0 T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right.$$
$$\left. + \quad C_0\omega_0 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2 + 4\epsilon^2 \sup_{T\in[0,T_0]} \|\nabla A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right].$$

By the Proposition 2.4,

$$\sup_{t\in[0,T_0/\epsilon^2]} Q_3 \quad \leq \quad 4\epsilon^{-2}\omega_0^2 T_0^2 e^{I_3 T_0 \epsilon^{-2}} \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left[\epsilon^{-2} C_0 T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right.$$
$$\left. + \quad C_0 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2 + 4\epsilon^2 \sup_{T\in[0,T_0]} \|\nabla A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right],$$

Taking into account (4.29), (4.39)-(4.43), from here, we deduce that

$$\delta_3 \leq \min\{\delta_1+\delta_2-4,\, 2\delta_2-2,\, \delta_2\}.$$

Because $\delta_1 = \frac{9}{2}$ and $\delta_2 = \frac{5}{2}$, we obtain

$$\delta_3 = \frac{5}{2}, \tag{4.49}$$

which is compatible with the condition $\delta_3 < \frac{13}{4}$ implied by (4.47).

Hence the third energy level is controlled and we close all the assumptions.

4.2.4 Proof of the Theorem 1.1

According to (4.29),(4.39), (4.49), we have the following estimates

$$\sup_{t\in[0,T_0/\epsilon^2]} \|V\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right), \quad \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla V\|_{L^2} = \mathcal{O}\left(\epsilon^{9/2}\right), \tag{4.50}$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|\Delta V\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right), \quad \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla \Delta V\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right), \tag{4.51}$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|M\|_{L^2} = \mathcal{O}\left(\epsilon^{9/2}\right), \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla M\|_{L^2} = \sup_{t\in[0,T_0/\epsilon^2]} \|\Delta M\|_{L^2} = \mathcal{O}\left(\epsilon^{5/2}\right). \tag{4.52}$$

Due to (4.11)-(4.12), this leads to the main final estimates in our justification analysis

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|E - \epsilon^2 (A)_{\omega_0}\right\|_{H^3(\mathbb{R}^2)} = \mathcal{O}\left(\epsilon^{5/2}\right), \tag{4.53}$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|m - \epsilon^2 m_0\right\|_{H^2(\mathbb{R}^2)} = \mathcal{O}\left(\epsilon^{5/2}\right). \tag{4.54}$$

Note that controllability of the right-hand sides of inequalities (4.27), (4.36), (4.48) is also due to zero initial conditions

$$\mathcal{H}_1\Big|_{t=0} = \mathcal{H}_2\Big|_{t=0} = \mathcal{H}_3\Big|_{t=0} = 0,$$

which hold because of exact match of initial data for approximated and true solutions.

Also, in the estimates throughout this section (in particular, those involving $R_8^{(V)}$, $R_8^{(M)}$, $R_6^{(M)}$, $R_4^{(M)}$) we assumed smoothness of $A(X,Z,T)$, $B(X,Z,T)$, $D(X,Z,T)$. This, however, follows from the Theorem 3.2 and Remark 3.2 providing the initial data $A_0(X,Z)$ is sufficiently smooth. Indeed, the most stringent requirements come from the estimates performed on the third energy level where we imposed conditions $\partial_X^2 A$, $\partial_Z^2 A$, $\partial_T^2 A \in L^\infty(\mathbb{R}^2)$ for all $T \in [0, T_0]$ and $\partial_X^2 R_8^{(V)}$, $\partial_Z^2 R_8^{(V)}$, $\partial_T^2 R_8^{(V)} \in L^2(\mathbb{R}^2)$ for all $t \in [0, T_0/\epsilon^2]$. Expressing T-derivatives from the equations (3.23), (4.6)-(4.7) and differentiating one more time with respect to T, we have to require $A_0 \in H^6(\mathbb{R}^2)$ from the L^∞-restriction, and $A_0 \in H^8(\mathbb{R}^2)$ from the $\partial_T^2 R_8^{(V)} \in L^2(\mathbb{R}^2)$ condition.

We have already obtained a bound for $\sup_{t\in[0,T_0/\epsilon^2]} \|U\|_{L^\infty}$ when performing estimates on the first energy level. Similarly, applying the Proposition 2.2 to the derivatives of (4.4) and using (4.50)-(4.51), we control

$$\sup_{t\in[0,T_0/\epsilon^2]} \|U\|_{L^\infty} = \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla U\|_{L^\infty} = \sup_{t\in[0,T_0/\epsilon^2]} \|U_t\|_{L^\infty} = \mathcal{O}\left(\epsilon^{5/2}\right). \tag{4.55}$$

This allows to apply the Theorem 3.1 to extend validity of local solution E up to time $t_0 = T_0/\epsilon^2$, and the estimates (4.53)-(4.54) furnish the justification of the NLS approximation.

The proof of the Theorem 1.1 is now complete.

Chapter 5

Numerical results

5.1 One-dimensional model

In this chapter, we illustrate our main results by numerical simulations. To simplify the setup for numerical work, we assume that all solutions are x-independent. This corresponds to the modulated pulse propagating in the z-direction which is uniform in the (x, y)-directions. Equations for the x-independent residual terms (1.10)-(1.11) become

$$\partial_z^2 U - \left(1 + \epsilon^2 m_0 + N\right) \partial_t^2 U = -\epsilon^2 \left(R_2^{(U)}\right)_{\omega_0} N - \epsilon^6 \left(R_6^{(U)}\right)_{\omega_0}, \tag{5.1}$$

$$\partial_t N = \epsilon^4 \left(A^2\right)_{2\omega_0} + 2\epsilon^2 \left(A\right)_{\omega_0} U + U^2, \tag{5.2}$$

subject to the zero initial conditions $U|_{t=0} = 0,\ \partial_t U|_{t=0} = 0,\ N|_{t=0} = 0$.

Note that $R_2^{(U)}$and $R_6^{(U)}$ are the same as defined in (1.12)-(1.13), whereas $A\,(Z, T)$ and $m_0\,(Z, T)$ solve the system (1.5)-(1.6) which becomes

$$\partial_Z A + \partial_T A = \frac{i\omega_0}{2} m_0 A, \tag{5.3}$$

$$\partial_T m_0 = 2\,|A|^2\,, \tag{5.4}$$

subject to the initial conditions $A\,(Z, 0) = A_0\,(Z)$, $m_0\,(Z, 0) = 0$.

The system (5.3)-(5.4) can be solved analytically by the method of characteristics.

Let $Z = z\,(s) = s + z_0$, $T = t\,(s) = s$ and denote $\mathcal{A}\,(s; z_0) := A\,(s + z_0, s)$, $\mathcal{M}\,(s; z_0) := m_0\,(s + z_0, s)$.

Integrating the second equation, we have

$$m_0\left(Z,T\right) = 2\int_0^T \left|A\left(Z,\tau\right)\right|^2 d\tau,$$

and hence

$$\mathcal{M}\left(s;z_0\right) = 2\int_0^s \left|A\left(s+z_0,\tau\right)\right|^2 d\tau = 2\int_0^s \left|\mathcal{A}\left(\tau;z_0-\tau+s\right)\right|^2 d\tau.$$

The first equation, then, becomes

$$\frac{d}{ds}\log\mathcal{A}\left(s;z_0\right) = i\omega_0\int_0^s \left|\mathcal{A}\left(\tau;z_0-\tau+s\right)\right|^2 d\tau.$$

Using polar decomposition $\mathcal{A}\left(s;z_0\right) = R\left(s;z_0\right)\exp\left(i\theta\left(s;z_0\right)\right)$, we obtain

$$R\left(s;z_0\right) = R\left(0;z_0\right),$$

and

$$\theta\left(s;z_0\right) = \theta\left(0;z_0\right) + \omega_0\int_0^s \left(\int_0^\lambda \left|A_0\left(z_0+\lambda-\tau\right)\right|^2 d\tau\right) d\lambda.$$

Finally, eliminating parameters $z_0 = Z-T$, $s=T$, we conclude

$$A\left(Z,T\right) = A_0\left(Z-T\right)\exp\left[i\omega_0\int_0^T \left(\int_{Z-T}^{Z-T+\lambda} \left|A_0\left(s\right)\right|^2 ds\right) d\lambda\right], \tag{5.5}$$

$$m_0\left(Z,T\right) = 2\int_{Z-T}^{Z} \left|A_0\left(s\right)\right|^2 ds. \tag{5.6}$$

5.2 Numerical set-up

We choose initial profile $A_0\left(Z\right) = \operatorname{sech}\left(Z\right)$. Then, according to (5.5)-(5.6),

$$A\left(Z,T\right) = \operatorname{sech}\left(Z-T\right)\exp\left[i\omega_0\left(\log\left(\frac{\cosh Z}{\cosh\left(Z-T\right)}\right) - T\tanh\left(Z-T\right)\right)\right], \tag{5.7}$$

$$m_0\left(Z,T\right) = 2\left(\tanh Z - \tanh\left(Z-T\right)\right). \tag{5.8}$$

With the following notation

$$\begin{aligned}
a &:= 1+\epsilon^2 m_0,\\
b &:= \epsilon^2\left[e^{i\omega_0(z-t)}\left(\omega_0^2 A + 2i\omega_0\epsilon^2\partial_T A - \epsilon^4\partial_T^2 A\right) + c.c.\right],\\
c &:= \epsilon^6\left[e^{i\omega_0(z-t)}\left(\partial_Z^2 A - \left(1+\epsilon^2 m_0\right)\partial_T^2 A + 2i\omega_0 m_0\partial_T A\right) + c.c.\right],\\
d &:= 2\epsilon^2\left(Ae^{i\omega_0(z-t)} + c.c.\right),\\
e &:= \epsilon^4\left(A^2 e^{2i\omega_0(z-t)} + c.c.\right).
\end{aligned}$$

the equations to be solved are

$$\partial_t^2 U = \frac{1}{a+N}\partial_z^2 U + \frac{bN+c}{a+N}, \tag{5.9}$$

$$\partial_t N = U^2 + dU + e. \tag{5.10}$$

Discretizing the computational domain, we denote the value of X at the spatial point i and the j-th time step by $X_i^{(j)}$.

We employ the two-step explicit Adams-Bashforth method for the equation (5.10) yielding

$$N_i^{(j+1)} = N_i^{(j)} + \frac{3}{2}\Delta t\left[\left(U_i^{(j)}\right)^2 + d_i^{(j)}U_i^{(j)} + e_i^{(j)}\right] - \frac{1}{2}\Delta t\left[\left(U_i^{(j-1)}\right)^2 + d_i^{(j-1)}U_i^{(j-1)} + e_i^{(j-1)}\right],$$

that is followed by application of the Crank-Nicholson type of scheme to solve the equation (5.9)

$$\begin{aligned}
\frac{U_i^{(j+1)} - 2U_i^{(j)} + U_i^{(j-1)}}{(\Delta t)^2} &= \frac{1}{2\left(a_i^{(j+1)} + N_i^{(j+1)}\right)}\left(\frac{U_{i+1}^{(j+1)} - 2U_i^{(j+1)} + U_{i-1}^{(j+1)}}{(\Delta z)^2} + b_i^{(j+1)}N_i^{(j+1)} + c_i^{(j+1)}\right)\\
&+ \frac{1}{2\left(a_i^{(j-1)} + N_i^{(j-1)}\right)}\left(\frac{U_{i+1}^{(j-1)} - 2U_i^{(j-1)} + U_{i-1}^{(j-1)}}{(\Delta z)^2} + b_i^{(j-1)}N_i^{(j-1)} + c_i^{(j-1)}\right).
\end{aligned}$$

Both methods have the second-order accuracy in both space and time. Initial conditions for U and N are approximated respectively using central differences scheme and Heun's predictor-corrector method:

$$U_i^{(0)} = U_i^{(1)} = 0,$$

$$N_i^{(0)} = 0, \quad N_i^{(1)} = \frac{\Delta t}{2}\left(e_i^{(0)} + e_i^{(1)}\right).$$

If the initial pulse starts at the center of the computational domain, by taking size of the domain large enough for the time of interest, we can assume zero boundary conditions for U at the ends of the interval in z.

5.3 Simulation results

The numerical simulation was performed for $\epsilon = 0.3$.

The Figures (1)-(12) present profiles of U and N versus z for various instances of t. We can observe the pulse grows in amplitude and broadens in width.

The Figures (13)-(16) show changes of the H^3- and L^∞-norms of the error terms U and N. The size of the error terms oscillates and slowly grows.

Repeating the simulation for different small values of the parameter ϵ, we obtain the data given in the Table 5.1. This allows us to estimate exponents in the power laws for the norms of the solution

$$\sup_{t\in[0,1]} \|U\|_{H^3(\mathbb{R})} = \mathcal{O}\left(\epsilon^{\hat{\alpha}}\right), \quad \sup_{t\in[0,1]} \|N\|_{H^2(\mathbb{R})} = \mathcal{O}\left(\epsilon^{\hat{\beta}}\right),$$

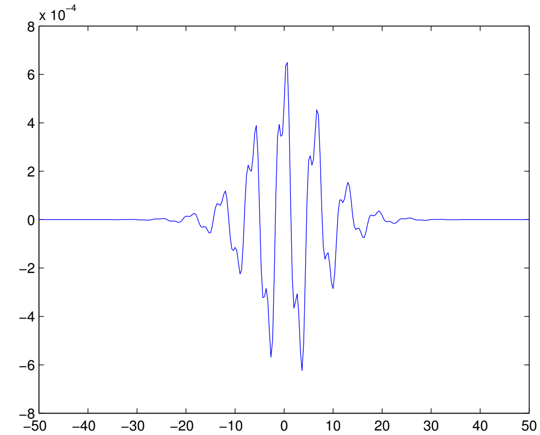
$$\sup_{t\in[0,1]} \|U\|_{L^\infty(\mathbb{R})} = \mathcal{O}\left(\epsilon^{\hat{\gamma}}\right), \quad \sup_{t\in[0,1]} \|N\|_{L^\infty(\mathbb{R})} = \mathcal{O}\left(\epsilon^{\hat{\delta}}\right).$$

We can conclude from the Table 5.1 that $\hat{\alpha} \approx 5$, $\hat{\beta} \approx 3$, $\hat{\gamma} \approx 6$, $\hat{\delta} \approx 4$. This is in agreement with the results (4.53)-(4.54).

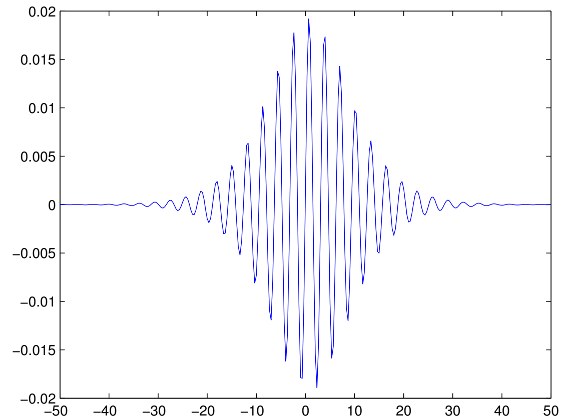
We note that the formal expansions (1.8) and (1.9) would suggest $\hat{\gamma} = \hat{\delta} = 4$, and the latter bound agrees with the numerical value $\hat{\delta} \approx 4$. The former bound is bigger than the actual numerical value $\hat{\gamma} \approx 6$ which may be due to some cancellations of error in our numerical simulations.

ϵ	0.40	0.30	0.20	0.15
$\|U\|_{H^3(\mathbb{R})}$	$6.86 \cdot 10^{-2}$	$1.67 \cdot 10^{-2}$	$9.50 \cdot 10^{-5}$	$2.26 \cdot 10^{-5}$
$\hat{\alpha}$	4.9113		4.9972	
$\|N\|_{H^2(\mathbb{R})}$	$41.09 \cdot 10^{-2}$	$17.09 \cdot 10^{-2}$	$3.25 \cdot 10^{-2}$	$1.37 \cdot 10^{-2}$
$\hat{\beta}$	3.0494		3.0028	
$\|U\|_{L^\infty(\mathbb{R})}$	$12.9 \cdot 10^{-3}$	$2.40 \cdot 10^{-3}$	$26.08 \cdot 10^{-6}$	$4.66 \cdot 10^{-6}$
$\hat{\gamma}$	5.8459		5.9879	
$\|N\|_{L^\infty(\mathbb{R})}$	$5.13 \cdot 10^{-2}$	$1.61 \cdot 10^{-2}$	$27.00 \cdot 10^{-4}$	$8.63 \cdot 10^{-4}$
$\hat{\delta}$	4.0283		3.9648	

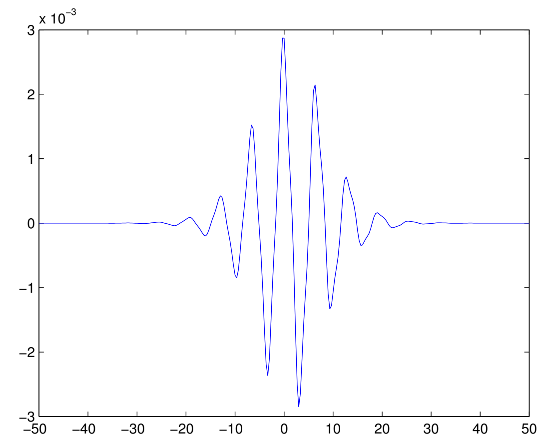
Table 5.1: Supremum of the solution norms over the time interval $\left[0, 1/\epsilon^2\right]$.



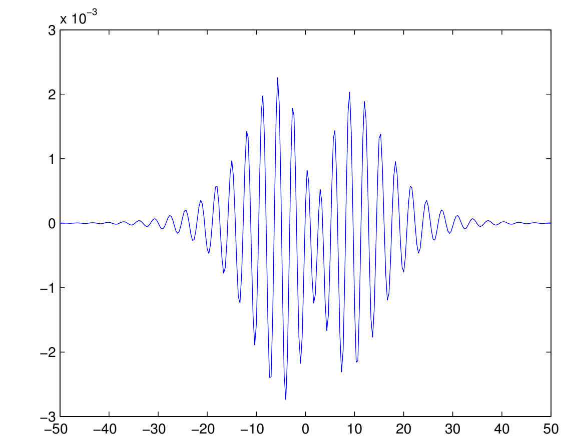
(1) $U(z)$ at $t = \epsilon^{-2}/6$



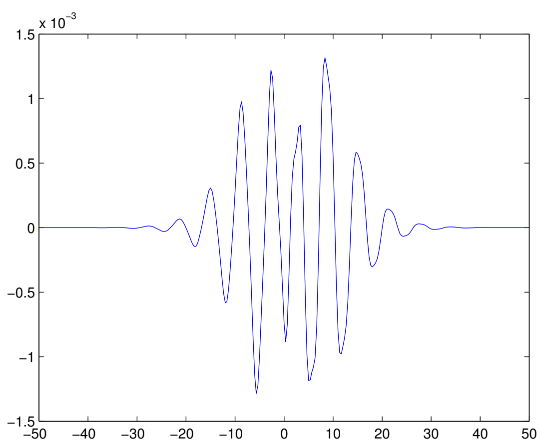
(2) $N(z)$ at $t = \epsilon^{-2}/6$



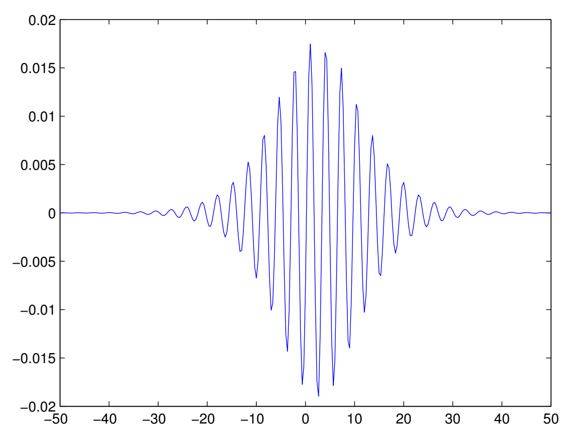
(3) $U(z)$ at $t = 2\epsilon^{-2}/6$



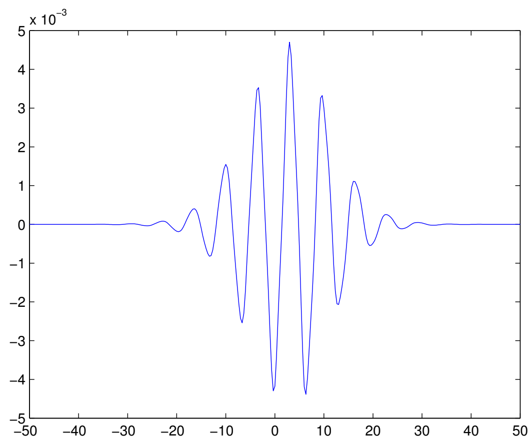
(4) $N(z)$ at $t = 2\epsilon^{-2}/6$



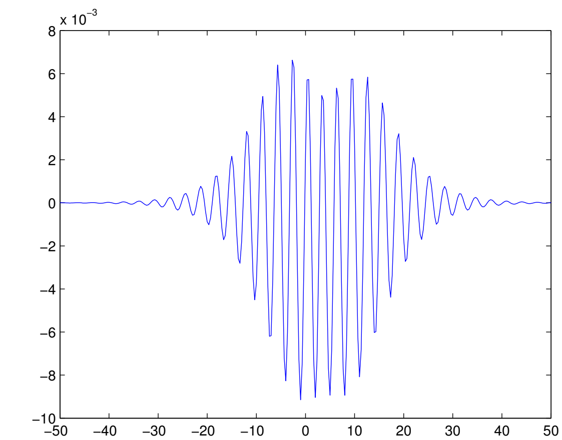
(5) $U(z)$ at $t = 3\epsilon^{-2}/6$



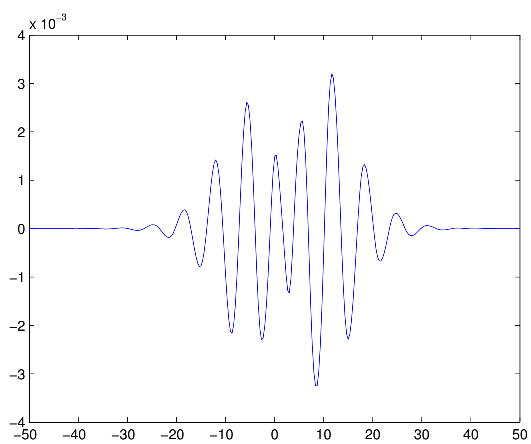
(6) $N(z)$ at $t = 3\epsilon^{-2}/6$



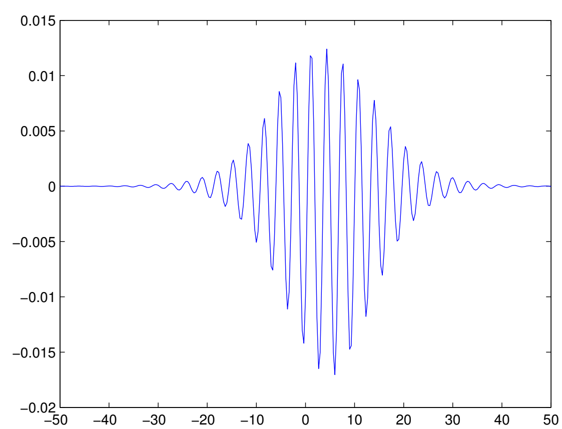
(7) $U(z)$ at $t = 4\epsilon^{-2}/6$



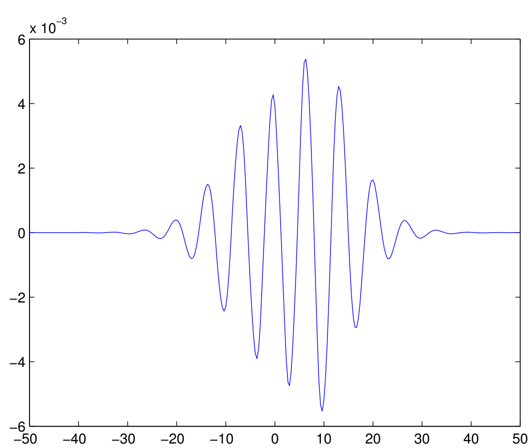
(8) $N(z)$ at $t = 4\epsilon^{-2}/6$



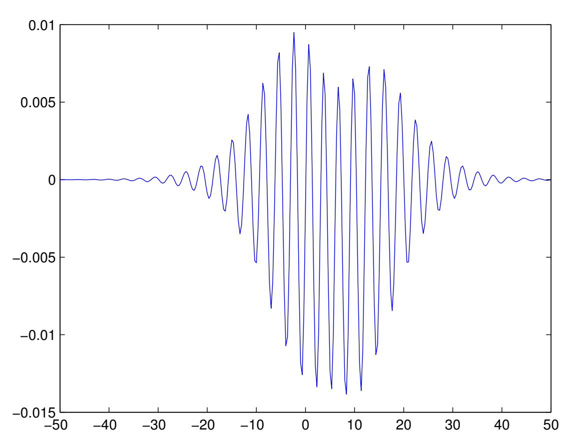
(9) $U(z)$ at $t = 5\epsilon^{-2}/6$



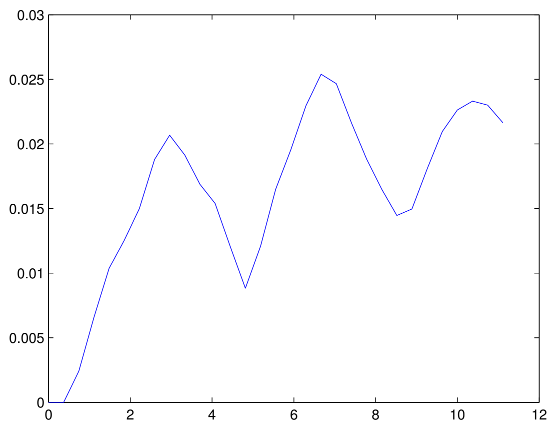
(10) $N(z)$ at $t = 5\epsilon^{-2}/6$



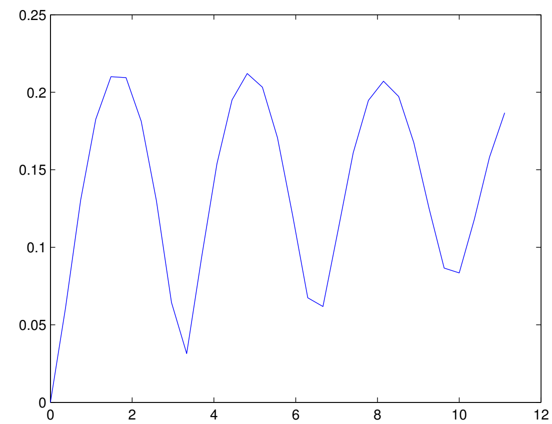
(11) $U(z)$ at $t = \epsilon^{-2}$



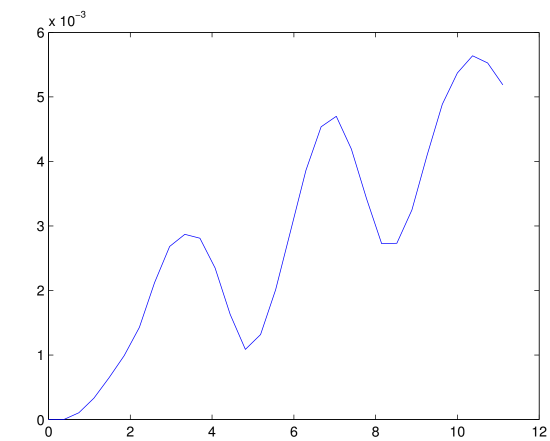
(12) $N(z)$ at $t = \epsilon^{-2}$



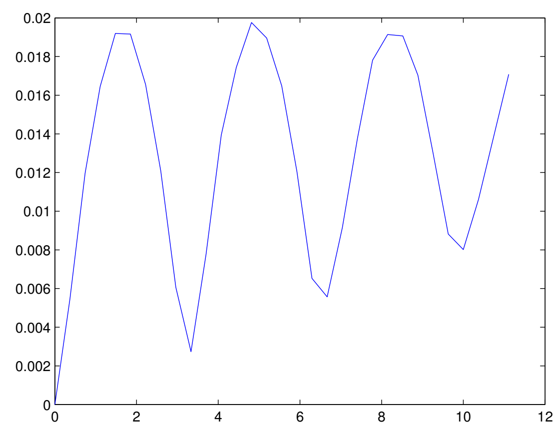
(13) $\|U\|_{H^3(\mathbb{R})}$ versus time t



(14) $\|N\|_{H^2(\mathbb{R})}$ versus time t



(15) $\|U\|_{L^\infty(\mathbb{R})}$ versus time t



(16) $\|N\|_{L^\infty(\mathbb{R})}$ versus time t

APPENDIX

Global justification for the linear system

Here, we demonstrate that the spatial Schrödinger equation used in the previous publications [8, 16] can be justified globally in the linear case, that is, when the squared refractive index $n^2 = 1$ is constant.

Due to the linearity of the problem, we can use separation of variables. In particular, we factor out time-dependence multiplier $e^{-i\omega_0 t}$ and, hence, work with the Helmholtz equation

$$\partial_x^2 E + \partial_z^2 E + \omega_0^2 E = 0, \quad x \in \mathbb{R},\, z \in \mathbb{R}_+, \tag{A.1}$$

subject to the given boundary condition $E|_{z=0} =: E_0(x) = A_0(\epsilon x)$ with small parameter $\epsilon > 0$.

The leading order approximation by the multiple-scale method reads

$$E_{ms}(x, z) = e^{i\omega_0 z} A(X, Z), \tag{A.2}$$

where $X := \epsilon x$, $Z := \epsilon^2 z$, and $A(X, Z)$ is governed by the linear Schrödinger equation

$$\partial_X^2 A + 2i\omega_0 \partial_Z A = 0, \qquad X \in \mathbb{R},\, Z \in \mathbb{R}_+ \tag{A.3}$$

with the boundary condition $A|_{Z=0} = A_0(X)$.

To estimate the validity of such approximation, we will take advantage of the fact that both the Helmholtz and the Schrödinger equations are exactly solvable with the Fourier transform. Once the solutions are found, it remains to compare them in the whole range of z.

Theorem A.1. *For any $s > 1/2$, let $A_0 \in H^s(\mathbb{R})$, $\epsilon > 0$. Then, if E solves (A.1), and E_{ms} is as defined by (A.2)-(A.3), the following estimate holds*

$$\sup_{z\in[0,Z_0/\epsilon^2]} \|E - E_{ms}\|_{L^\infty(\mathbb{R})} = \mathcal{O}\left(\epsilon^{\frac{2(2s-1)}{2s+7}}\right). \tag{A.4}$$

Moreover, under additional assumptions on the Fourier transform of the boundary data, $\tilde{A}_0 \in L^\infty(\mathbb{R})$, $\tilde{A}_0' \in L^1(\mathbb{R})$, we have

$$\sup_{z\in\mathbb{R}_+} \|E - E_{ms}\|_{L^\infty(\mathbb{R})} = \mathcal{O}\left(\epsilon^{\frac{2(2s-1)}{6s+5}}\right). \tag{A.5}$$

Proof. Using Fourier transform in the slow variable X, we write the solution to (A.3) in the form

$$A(X,Z) = \int_{\mathbb{R}} \tilde{A}_0(k) \exp\left[-ikX - i\frac{k^2}{2\omega_0}Z\right] dk,$$

where

$$\tilde{A}_0(k) := \frac{1}{2\pi}\int_{\mathbb{R}} A_0(X) e^{ikX} dX.$$

Similarly, taking into account radiation (when $\xi < \omega_0$) and an exponential decay (when $\xi > \omega_0$) conditions as $z \to \infty$, the solution to the equation (A.1) is obtained in the form

$$E(x,z) = \int_{\mathbb{R}} \hat{E}_0(\xi) \exp\left[-i\xi x + i\sqrt{\omega_0^2 - \xi^2} z\right] d\xi,$$

where

$$\hat{E}_0(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}} E_0(x) e^{i\xi x} dx = \epsilon^{-1}\tilde{A}_0\left(\epsilon^{-1}\xi\right).$$

By the duality $x \leftrightarrow \xi$, $X \leftrightarrow k$, we have $k = \xi/\epsilon$.

Using (A.2), we introduce the error term $(E - E_{ms})$ that shall be controlled in L^∞-norm

$$E - E_{ms} = \int_{\mathbb{R}} \hat{E}_0(\xi) e^{-i\xi x}\left(\exp\left[-i\omega_0 z + i\sqrt{\omega_0^2 - \xi^2} z\right] - \exp\left[-i\frac{\xi^2 z}{2\omega_0}\right]\right) d\xi.$$

We proceed in two steps. First, we control $\|E - E_{ms}\|_{L^\infty}$ for extended range $0 \le z \le \frac{Z_0}{\epsilon^{2+\mu}}$

for $\mu>0$, whereas the case $\mu=0$ will correspond to the local justification result (A.4). Then, for $\lambda>0$, we control $\|E-E_{ms}\|_{L^\infty}$ for $z\geq\frac{Z_0}{\epsilon^{2+\lambda}}$ as well. Choosing the exponents μ and λ such that the both ranges overlap, that is $\mu\geq\lambda$, we will deduce the global justification result (A.5).

In the estimates below, C denotes a positive generic constant which value can change from line to line.

Let us fix some $\delta\in(0,1)$, $\mu>0$ and estimate

$$\begin{aligned}\sup_{0\leq z\leq\frac{Z_0}{\epsilon^{2+\mu}}}\|E-E_{ms}\|_{L^\infty(\mathbb{R})} &\leq \sup_{0\leq z\leq\frac{Z_0}{\epsilon^{2+\mu}}}\int_{\mathbb{R}}\left|\hat{E}_0(\xi)\right|\cdot\left|\exp\left(-i\omega_0 z+i\sqrt{\omega_0^2-\xi^2}z\right)\right. \\ &\quad \left.-\exp\left(-i\frac{\xi^2 z}{2\omega_0}\right)\right|d\xi=\sup_{0\leq z\leq\frac{Z_0}{\epsilon^{2+\mu}}}\left[\int_{|\xi|\leq\epsilon^{1-\delta}}+\int_{|\xi|\geq\epsilon^{1-\delta}}\right] \\ &=: J_1+J_2. \end{aligned}\tag{A.6}$$

Setting

$$\theta(\xi):=\sqrt{\omega_0^2-\xi^2}-\omega_0+\frac{\xi^2}{2\omega_0},$$

we bound the first term in (A.6), for any $s>1/2$,

$$\begin{aligned}J_1 &\leq \sup_{0\leq z\leq\frac{Z_0}{\epsilon^{2+\mu}}}\int_{|\xi|\leq\epsilon^{1-\delta}}\left|\hat{E}_0(\xi)\right|\cdot\left|e^{i\theta(\xi)z}-1\right|d\xi\lesssim\frac{1}{(2\omega_0)^3}\sup_{0\leq z\leq\frac{Z_0}{\epsilon^{2+\mu}}}\epsilon^{4(1-\delta)}z\int_{|\xi|\leq\epsilon^{1-\delta}}\left|\hat{E}_0(\xi)\right|d\xi \\ &= \frac{Z_0}{(2\omega_0)^3}\epsilon^{2(1-2\delta)-\mu}\int_{|k|\leq\epsilon^{-\delta}}\left|\tilde{A}_0(k)\right|dk\leq C\epsilon^{2(1-2\delta)-\mu}\left(\int_{|k|\leq\epsilon^{-\delta}}\left|\tilde{A}_0(k)\right|^2\left(1+k^2\right)^s dk\right)^{1/2} \\ &\leq C\epsilon^{2-4\delta-\mu}\|A_0\|_{H^s},\end{aligned}$$

where we used the Cauchy-Schwarz inequality, Plancherel theorem and the bound for small values of ξ

$$\left|e^{i\theta(\xi)z}-1\right|=2\left|\sin\left(\frac{\theta(\xi)z}{2}\right)\right|\leq|\theta(\xi)z|\lesssim\frac{\xi^4}{(2\omega_0)^3}z.$$

Hence, J_1 is small if $\mu<2(1-2\delta)$ and since $\mu>0$, we must have $\delta<\frac{1}{2}$. Also, clearly, $\mu<2$.

Now, we proceed with the second term in (A.6). We use regularity of A_0 to obtain desired smallness,

$$\begin{aligned} J_2 &\leq 2\int_{|\xi|\geq\epsilon^{1-\delta}} \left|\hat{E}_0(\xi)\right| d\xi = 2\int_{|k|\geq\epsilon^{-\delta}} \left|\tilde{A}_0(k)\right| dk \\ &\leq C\|A_0\|_{H^s} \left(\int_{|k|\geq\epsilon^{-\delta}} \frac{dk}{(1+k^2)^s}\right)^{1/2} \leq C\|A_0\|_{H^s}\,\epsilon^{(2s-1)\delta/2}, \end{aligned}$$

for any $s > \frac{1}{2}$.

Therefore,

$$\sup_{0\leq z\leq \frac{Z_0}{\epsilon^{2+\mu}}} \|E - E_{ms}\|_{L^\infty(\mathbb{R})} = \mathcal{O}(\epsilon^\alpha), \tag{A.7}$$

with $\alpha = \min\{2 - 4\delta - \mu,\ (2s-1)\delta/2\}$, and hence, to maximize the smallness power α, we choose $\delta = \frac{2(2-\mu)}{2s+7}$ yielding

$$\alpha = \frac{(2-\mu)(2s-1)}{2s+7}. \tag{A.8}$$

In particular, for $\mu = 0$, we obtain the local justification result (A.4).

Note that there was no restriction on the z-interval in J_2. The restriction comes when integrating in ξ near the origin. To improve this, we take advantage of the oscillatory factor $e^{-i\xi x}$ to obtain bound for larger values of z

$$\begin{aligned} \sup_{z\geq\frac{Z_0}{\epsilon^{2+\lambda}}} \|E - E_{ms}\|_{L^\infty(\mathbb{R})} \leq \sup_{z\geq\frac{Z_0}{\epsilon^{2+\lambda}}} \sup_{x\in\mathbb{R}} &\left[\left|\int_{|k|\leq\epsilon^{-\delta}} \tilde{A}_0(k)\exp\left(-i\epsilon kx - i\omega_0 z + i\sqrt{\omega_0^2-\epsilon^2k^2}\,z\right) dk\right|\right. \\ &+ \left.\left|\int_{|k|\leq\epsilon^{-\delta}} \tilde{A}_0(k)\exp\left(-i\epsilon kx - i\frac{\epsilon^2k^2z}{2\omega_0}\right) dk\right|\right], \quad \lambda > 0. \end{aligned}$$

Setting

$$\phi(k) := -\frac{kx}{\epsilon z} - \frac{\omega_0}{\epsilon^2} + \sqrt{\frac{\omega_0^2}{\epsilon^4} - \frac{k^2}{\epsilon^2}},$$

we have $|\phi''(k)| = \frac{\omega_0^2}{\left(\omega_0^2-\epsilon^2k^2\right)^{3/2}} \geq C > 0$ for $|k| \leq \epsilon^{-\delta}$ since $\delta < 1$. This allows to apply the van der Corput lemma (see, for instance, [6]) to estimate the first integral as

$$\sup_{z\geq\frac{Z_0}{\epsilon^{2+\lambda}}}\left|\int_{|k|\leq\epsilon^{-\delta}}\tilde{A}_0\left(k\right)\exp\left(\phi\left(k\right)\epsilon^2 z\right)dk\right|\leq C\epsilon^{\lambda/2}\left(\left\|\tilde{A}_0\right\|_{L^\infty(\mathbb{R})}+\left\|\tilde{A}_0'\right\|_{L^1(\mathbb{R})}\right).$$

Clearly, in the same fashion we can bound the second integral, so we finally have

$$\sup_{z\geq\frac{Z_0}{\epsilon^{2+\lambda}}}\|E-E_{ms}\|_{L^\infty(\mathbb{R})}=\mathcal{O}\left(\epsilon^{\lambda/2}\right).$$

Summing up the results, we obtain

$$\sup_{z\in\mathbb{R}_+}\|E-E_{ms}\|_{L^\infty(\mathbb{R})}=\mathcal{O}\left(\epsilon^{\beta}\right),$$

where

$$\beta:=\min\left\{\alpha,\,\frac{\lambda}{2}\right\}=\min\left\{\frac{(2-\mu)\,(2s-1)}{2s+7},\,\frac{\lambda}{2}\right\}.$$

We require $\lambda\leq\mu$ to have overlap between intervals $0\leq z\leq\frac{Z_0}{\epsilon^{2+\mu}}$ and $z\geq\frac{Z_0}{\epsilon^{2+\lambda}}$. Taking $\mu=\frac{4(2s-1)}{6s+5}$, we obtain $\beta=\frac{2(2s-1)}{6s+5}$ which provides the optimal error bound in the performed analysis completing the proof of the estimate (A.5). □

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