Magnetization moment recovery based on Kelvin transformation and Fourier analysis

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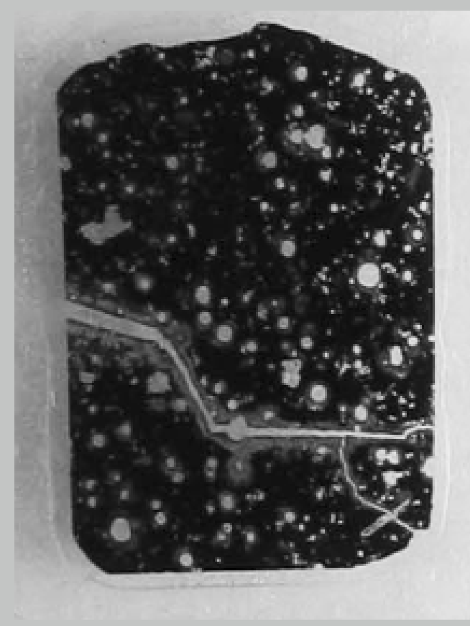
Introduction

SQUID microscope (EAPS lab, MIT)

Some ancient georocks and meteorites possess remanent magnetization and thus store valuable records of the past magnetic field on Earth and other planets or satellites. Thanks to recent advances in magnetometry (SQUID microscope technique) offering a possibility to measure magnetic fields of very low intensities, extraction of this relict magnetic information has become reality. An endeavor to develop a robust and efficient method for processing these data leads to a number of challenging problems such as effective extension of the restricted measurement data and extraction of certain features of the magnetization (typically, its mean value) without solving the entire inverse problem. In the present work, we outline derivation of explicit formulas for estimation of the net magnetization moment vector of the sample in terms of the vertical component of the magnetic field measured in the plane above it.



Slice of a magnetized sample (basalt)

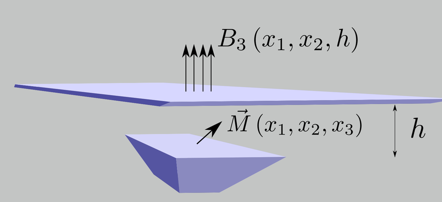
1. Problem formulation

Magnetization of the sample is an unknown function/distribution $\vec{M}(\mathbf{x},x_3) = (M_1(\mathbf{x},x_3), M_2(\mathbf{x},x_3), M_3(\mathbf{x},x_3))^T$, $\mathbf{x} := (x_1,x_2)^T$, $\vec{x} := (\mathbf{x},x_3)$ supported on a subset $Q \subset \mathbb{R}^3$.
Representing the magnetic field in terms of the scalar potential Φ yields

$$\vec{B} = -\nabla\Phi + \vec{M} \quad \Rightarrow \quad \Delta\Phi = \nabla\cdot\vec{M},$$

and so

$$\Phi(\mathbf{x},x_3) = -\frac{1}{4\pi}\iiint_{\mathbb{R}^3} \frac{\nabla\cdot\vec{M}(\mathbf{t},t_3)}{\left(|\mathbf{x}-\mathbf{t}|^2 + (x_3-t_3)^2\right)^{1/2}} d^3t.$$



Geometry of the model

Experimentally, the vertical component of the magnetic field produced by the magnetic sample

$$B_3(\mathbf{x},h) = \frac{1}{4\pi}\iiint_Q \frac{3(h-t_3)\left[M_1(\mathbf{t},t_3)(x_1-t_1) + M_2(\mathbf{t},t_3)(x_2-t_2)\right] + M_3(\mathbf{t},t_3)\left(2(h-t_3)^2 - |\mathbf{x}-\mathbf{t}|^2\right)}{\left(|\mathbf{x}-\mathbf{t}|^2 + (h-t_3)^2\right)^{5/2}} d^3t$$

is measured on a part of the horizontal plane at height $x_3 = h > 0$.

- Goal: Knowing $B_3(\mathbf{x},h)$ or $\Phi(\mathbf{x},h)$ on $\mathbb{R}^2$ or $D_A := \left\{\mathbf{x}\in\mathbb{R}^2 : |\mathbf{x}| \le A\right\}$, we want to estimate the net magnetization moment of the sample

  $\vec{m} := \iiint_Q \vec{M}(\vec{x})\, d^3x \in \mathbb{R}^3$ or, more generally, $\left\langle M_i x_{k_1}^{j_1} x_{k_2}^{j_2}\right\rangle := \iiint_Q M_i(\vec{x})\, x_{k_1}^{j_1} x_{k_2}^{j_2} d^3x \in \mathbb{R}$

  for $i, k_1, k_2 \in \{1,2,3\}$, $j_1, j_2 \in \{0,1,2\}$.

3. Incomplete data and Fourier analysis

In practice, we have only partial measurements of the field. Without loss of generality, we assume the measurement area to be the disk, i.e. $B_3(\mathbf{x},h)$ is available for $\mathbf{x} \in D_A$.

Asymptotic continuation of partial data

From asymptotical behavior of $B_3(\mathbf{x},h)$ for large $\mathbf{x}$, using symmetry, $\iint_{\mathbb{R}^2\setminus D_A} B_3(\mathbf{x},h)\,dx_1dx_2 = \frac{m_3}{2A} + \mathcal{O}\left(\frac{1}{A^3}\right)$.
Combining this with implication of Gauss theorem,

$$0 = \iint_{\mathbb{R}^2} B_3(\mathbf{x},h)\,dx_1dx_2 = \iint_{D_A} B_3(\mathbf{x},h)\,dx_1dx_2 + \iint_{\mathbb{R}^2\setminus D_A} B_3(\mathbf{x},h)\,dx_1dx_2,$$

we can estimate for the normal component of net moment

$$m_3 = 2A\iint_{D_A} B_3(\mathbf{x},h)\,dx_1dx_2 + \mathcal{O}\left(\frac{1}{A^2}\right).$$

Performing the same asymptotical extension of the measurements in the previously obtained integral formulas, we obtain estimates for the tangential components of the net moment

$$m_i = 2\iint_{D_A} B_3(\mathbf{x},h)\, x_i dx_1dx_2 + \mathcal{O}\left(\frac{1}{A}\right), \qquad i \in \{1,2\}.$$

Analysis in the Fourier domain

Define Fourier transform as $\mathcal{F}[f](\mathbf{k}) = \hat{f}(\mathbf{k}) = \iint_{\mathbb{R}^2} f(\mathbf{x})\, e^{2\pi i\mathbf{k}\cdot\mathbf{x}} dx_1dx_2$, $\mathbf{k} := (k_1,k_2)^T$, and observe that $\mathcal{F}\left[\left(|\mathbf{x}-\mathbf{t}|^2 + H^2\right)^{-\frac{3}{2}}\right](\mathbf{k}) = \frac{2\pi}{H} e^{-2\pi H|\mathbf{k}|}$ for $H > 0$. Then, representing the field as

$$B_3(\mathbf{x},h) = -\frac{1}{4\pi}\iiint_Q \left[(h-t_3)\left(M_1(\mathbf{t},t_3)\left.\frac{\partial}{\partial x_1}\right|_{x_3=h} + M_2(\mathbf{t},t_3)\left.\frac{\partial}{\partial x_2}\right|_{x_3=h}\right) + M_3(\mathbf{t},t_3)\left.\frac{\partial}{\partial x_3}\right|_{x_3=h}(x_3-t_3)\right]\left(|\vec{x}-\vec{t}|^2\right)^{-\frac{3}{2}} d^3t$$

and denoting Q_3 the vertical projection of the magnetization support set Q,

$$\hat{B}_3(\mathbf{k},h) = \pi\int_{Q_3} e^{-2\pi(h-t_3)|\mathbf{k}|}\left[ik_1\hat{M}_1(\mathbf{k},t_3) + ik_2\hat{M}_2(\mathbf{k},t_3) + |\mathbf{k}|\,\hat{M}_3(\mathbf{k},t_3)\right] dt_3.$$

Note that $\vec{m} = \int_{Q_3} \hat{\vec{M}}(\mathbf{0},t_3)\, dt_3$ and $\hat{\vec{M}}(\mathbf{k},t_3)$ is analytic in $\mathbf{k}$. So we expand about $\mathbf{k} = \mathbf{0}$ all the right-hand side terms as well as the first integral in $\hat{B}_3(\mathbf{k},h) = \left(\iint_{D_A} + \iint_{\mathbb{R}^2\setminus D_A}\right) e^{2\pi i\mathbf{k}\cdot\mathbf{x}} B_3(\mathbf{x},h)\,dx_1dx_2$ and perform asymptotic continuation in the second one. Matching terms of different smallness in $\mathbf{k}$, we obtain a set of relations whose combination yields higher-accuracy formulas for the tangential net moment components

$$m_i = 2\iint_{D_A}\left(1 + \frac{4x_i^2}{3A^2}\right) x_i B_3(\mathbf{x},h)\,dx_1dx_2 + \mathcal{O}\left(\frac{1}{A^2}\right), \qquad i \in \{1,2\}.$$

References

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2. Kelvin transform approach

Motivation for the method

Let us pretend that we have potential data on a sphere of radius $r = R_0$ encompassing the sample, i.e. the left-hand side of

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi}\iiint_Q \frac{M_1(\mathbf{t},t_3)(r\sin\theta\cos\phi - t_1) + M_2(\mathbf{t},t_3)(r\sin\theta\sin\phi - t_2) + M_3(\mathbf{t},t_3)(r\cos\theta - t_3)}{\left(r^2 - 2r\left[(t_1\cos\phi + t_2\sin\phi)\sin\theta + t_3\cos\theta\right] + t_1^2 + t_2^2 + t_3^2\right)^{3/2}} d^3t.$$

By harmonicity in the exterior of the sphere, we expand over spherical harmonics

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\sum_{j=-l}^{l} c_{j,l} S_l^j(\theta,\phi), \quad S_l^j(\theta,\phi) := \begin{cases} P_l^j(\cos\theta)\cos(j\phi), & j \ge 0, \\ P_l^{|j|}(\cos\theta)\sin(|j|\phi), & j < 0, \end{cases}$$

and observe that $c_{0,0} = 0$ and

$$\lim_{R\to\infty}\left\langle \Phi, \left(S_1^{-1}, S_1^0, S_1^1\right)^T\right\rangle_{L^2(\mathbb{S}_R)} = \left(-\frac{1}{3}m_2, \frac{1}{3}m_3, -\frac{1}{3}m_1\right)^T = \left(\frac{4\pi}{3}c_{-1,1}, \frac{4\pi}{3}c_{0,1}, \frac{4\pi}{3}c_{1,1}\right)^T.$$

Then, we easily retrieve the net moment:

$$m_1 = -3\left\langle \Phi, S_1^1\right\rangle_{L^2(\mathbb{S}_{R_0})}, \quad m_2 = -3\left\langle \Phi, S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})}, \quad m_3 = 3\left\langle \Phi, S_1^0\right\rangle_{L^2(\mathbb{S}_{R_0})}.$$

However, we have the data on the plane rather than on a sphere.

Kelvin transformation

Recall the Möbius transform $\frac{z-i}{z+i}$ which maps upper half-plane to the unit disk preserving harmonicity. 3D generalization of this is Kelvin transform $\mathcal{K}[f]\left(\vec{\xi}\right) = f^\star\left(\vec{\xi}\right) = \frac{1}{\left|\vec{\xi}-\vec{s}\right|} f\left(\mathcal{R}\vec{\xi}\right)$, where $\vec{\xi} := (\xi_1,\xi_2,\xi_3)^T$,

$$\mathcal{R}\vec{\xi} := \left(\frac{e_0^2\xi_1}{\xi_1^2+\xi_2^2+(\xi_3+R_0)^2}, \frac{e_0^2\xi_2}{\xi_1^2+\xi_2^2+(\xi_3+R_0)^2}, -R_0 + \frac{e_0^2(\xi_3+R_0)}{\xi_1^2+\xi_2^2+(\xi_3+R_0)^2}\right)^T, \quad \vec{s} := (0,0,-R_0)^T, \quad e_0 := \sqrt{2R_0(R_0+h)}.$$

Then: $\quad \Delta f(\mathbf{x},x_3) = 0, \quad x_3 > h \quad \Longleftrightarrow \quad \Delta f^\star\left(\vec{\xi}\right) = 0, \quad \left|\vec{\xi}\right| < R_0.$

Application of this transform to the potential gives

$$\mathcal{K}[\Phi](\theta,\phi) = \iiint_Q \frac{M_1(\mathbf{t},t_3)\left(\frac{(R_0+h)\sin\theta\cos\phi}{1+\cos\theta} - t_1\right) + M_2(\mathbf{t},t_3)\left(\frac{(R_0+h)\sin\theta\sin\phi}{1+\cos\theta} - t_2\right) + M_3(\mathbf{t},t_3)(h-t_3)}{4\pi R_0\sqrt{2(1+\cos\theta)}\left[\left(\frac{(R_0+h)\sin\theta\cos\phi}{1+\cos\theta} - t_1\right)^2 + \left(\frac{(R_0+h)\sin\theta\sin\phi}{1+\cos\theta} - t_2\right)^2 + (h-t_3)^2\right]^{3/2}} d^3t.$$

Unfortunately, because of more complicated angular dependence,

$$\left\langle \mathcal{K}[\Phi], S_1^1\right\rangle_{L^2(\mathbb{S}_{R_0})} \nsim m_1, \quad \left\langle \mathcal{K}[\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} \nsim m_2, \quad \left\langle \mathcal{K}[\Phi], S_1^0\right\rangle_{L^2(\mathbb{S}_{R_0})} \nsim m_3.$$

However, it is still true that $\lim_{R_0\to\infty}\left\langle \mathcal{K}[\Phi], S_1^1\right\rangle_{L^2(\mathbb{S}_{R_0})} \sim m_1$, $\lim_{R_0\to\infty}\left\langle \mathcal{K}[\Phi], S_1^{-1}\right\rangle_{L^2(\mathbb{S}_{R_0})} \sim m_2$, namely,

$$m_i = 6\lim_{R_0\to\infty} R_0^4 \iint_{\mathbb{R}^2} \Phi(\mathbf{x},h)\frac{x_i}{\left[x_1^2+x_2^2+(R_0+h)^2\right]^{5/2}} dx_1dx_2, \quad i \in \{1,2\}.$$

Employing spherical harmonics expansion and a specially derived connection formula

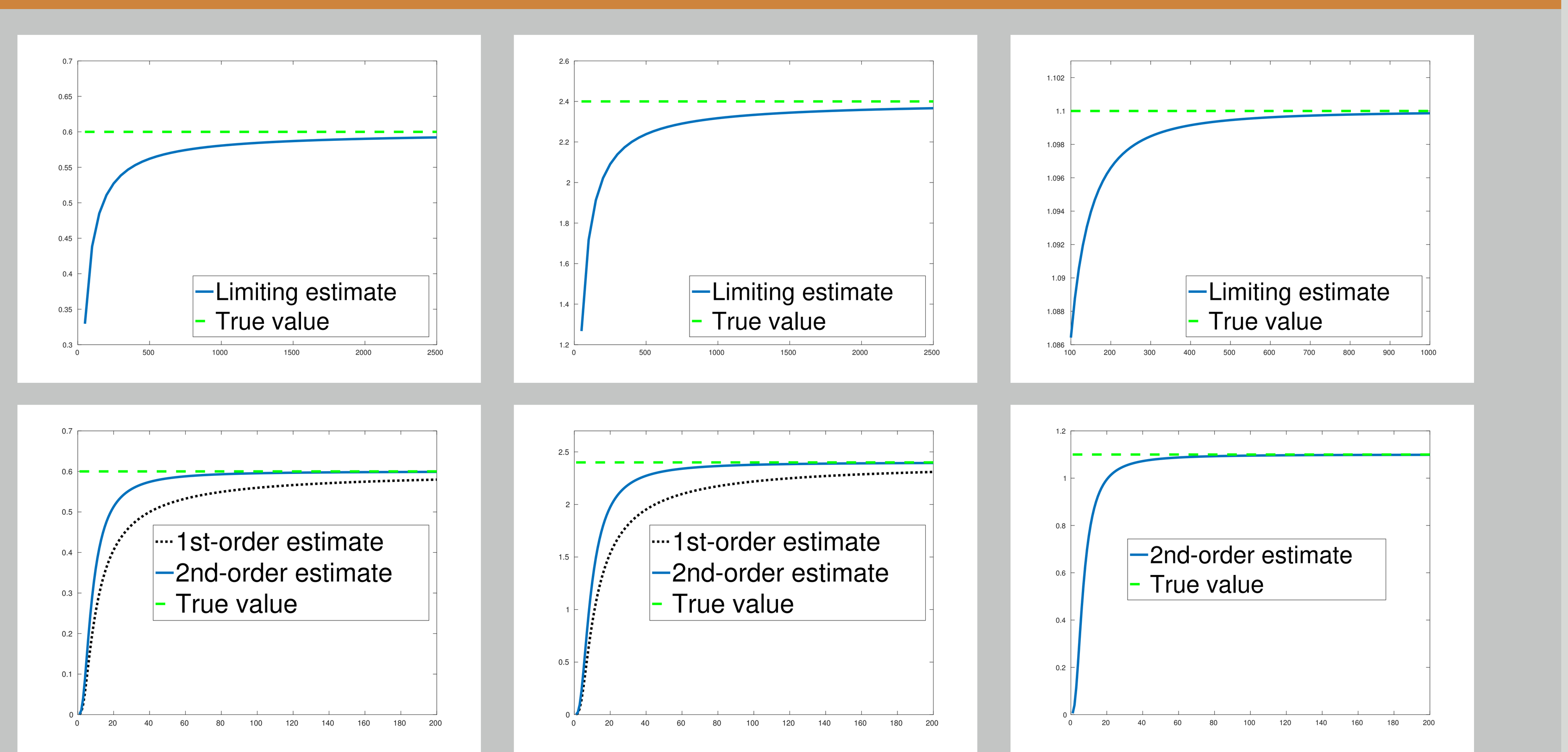
$$\mathcal{K}[\partial_{x_3}\Phi]\left(\vec{\xi}\right) = -\frac{1}{e_0^2}(R_0+\xi_3)\left(\mathcal{K}[\Phi]\left(\vec{\xi}\right) + 2R_0\partial_r\mathcal{K}[\Phi]\left(\vec{\xi}\right)\right), \quad \vec{\xi} \in \mathbb{S}_{R_0}, \quad \text{we obtain}$$

$$m_i = 2\lim_{R_0\to\infty} R_0^5 \iint_{\mathbb{R}^2} B_3(\mathbf{x},h)\frac{x_i}{\left[x_1^2+x_2^2+(R_0+h)^2\right]^{5/2}} dx_1dx_2, \quad i \in \{1,2\}.$$

Recovery of the normal component goes another way: it essentially stems from Poisson representation formula

$$m_3 = -2\lim_{\rho\to\infty}\rho^3\int_0^{2\pi} B_3(\rho\cos\varphi, \rho\sin\varphi, h)\, d\varphi.$$

4. Numerical illustrations of the obtained formulas



Recovery of m_1-m_3 (left to right) for the case of complete data (top row; very large fixed A; limiting variable on the horizontal axis) and partial data (bottom row; A varies along the horizontal axis)

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