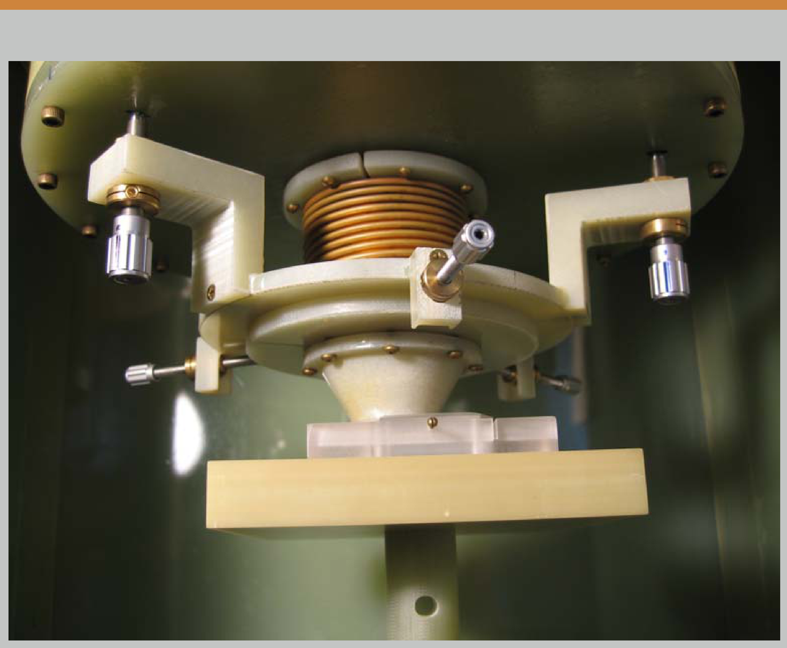
On some constructive aspects of an inverse problem in paleomagnetism

Eduardo Andrade Lima[1], Dmitry Ponomarev & FACTAS team[2]

[1]Paleomagnetism lab, EAPS department, MIT (USA); [2]Centre Inria d'Université Côte d'Azur (France).

Introduction



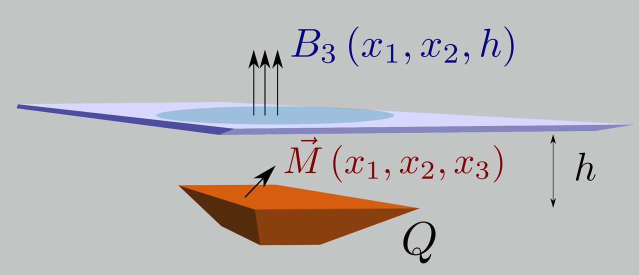
SQUID microscope (EAPS, MIT)

The process of extraction of relict magnetic information from georocks and meteorites is a challenging task. Due to the weak intensity of the field produced by the remanent magnetisation of a rock, the measurements have to be performed in direct vicinity of a sample and using highly sensitive magnetometric devices such as SQUID and QDM. Full reconstruction of the magnetic distribution (3D vector field) from partial measurements of only one component of the magnetic field is difficult and often replaced by estimation of the net magnetisation of a sample. We show some explicit asymptotic results for the latter problem, but observe that the lack of measured data and the presence of noise still remain an issue. We are thus motivated to consider the problem of stable extrapolation of magnetic field measurements.

Slice of a magnetised sample (basalt)

1. Setting

- Unknown sample magnetisation:
  $\vec{\mathcal{M}}(\vec{x}) \equiv (\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)(\boldsymbol{x}, x_3)$,
  with $\operatorname{supp} \vec{\mathcal{M}} \subset Q$, $|Q| < \infty$,
  $\boldsymbol{x} \equiv [x_1, x_2]^T$, $\vec{x} \equiv [\boldsymbol{x}, x_3]^T$.



Geometry of the model

- Measured component of the magnetic field:

$$B_3(\boldsymbol{x}, h) = \frac{1}{4\pi}\frac{\partial}{\partial h}\iiint_Q \frac{\nabla \cdot \vec{\mathcal{M}}(\boldsymbol{t}, t_3)}{\left(|\boldsymbol{x}-\boldsymbol{t}|^2 + (h-t_3)^2\right)^{1/2}} \mathrm{d}^3\vec{t}, \quad \boldsymbol{x} \in \mathbb{R}^2.$$

$$\left[\quad \vec{B} = -\nabla\Phi \quad \text{outside } Q, \qquad \Delta\Phi = \nabla\cdot\vec{\mathcal{M}} \quad \text{in } \mathbb{R}^3 \quad \right]$$

- Recovery $\vec{\mathcal{M}}(\vec{x}) \leftarrow B_3(\boldsymbol{x}, h)$ is a severely ill–posed problem:
  - The range of the mapping $\nabla\cdot\vec{\mathcal{M}} \longmapsto B_3$ is not closed.
  - The mapping $\vec{\mathcal{M}} \longmapsto \nabla\cdot\vec{\mathcal{M}}$ is not injective.
- Net magnetisation $\vec{m} := \iiint_Q \vec{\mathcal{M}}(\vec{x})\,\mathrm{d}^3\vec{x}$ is unique and important.

2. Net magnetisation: asymptotic results

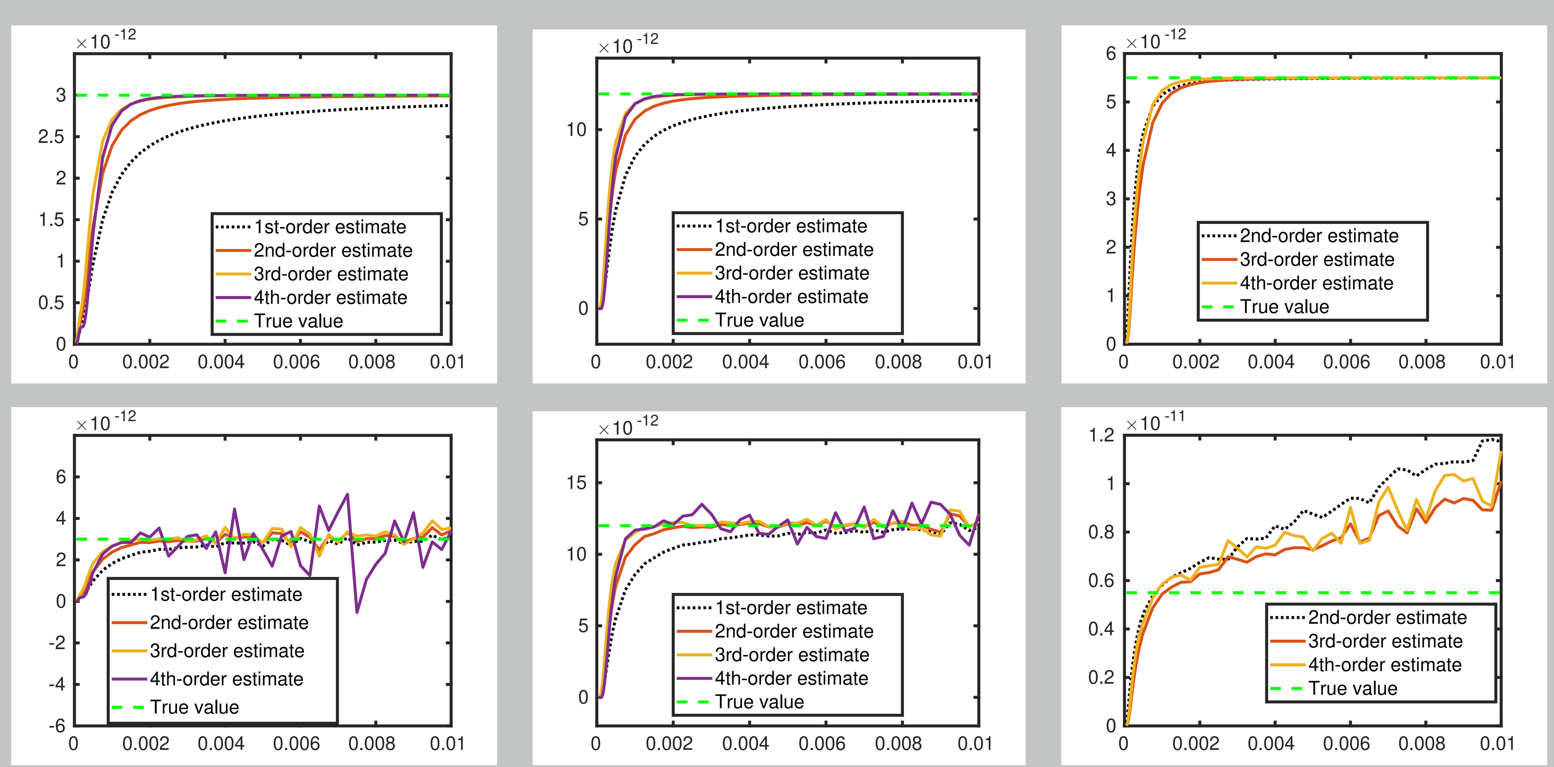
We can asymptotically estimate (explicitly and to an arbitrary order)

$$\vec{m} \leftarrow B_3(\cdot, h)|_{D_A}, \qquad D_A := \left\{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| < A\right\}, \qquad A \gg 1.$$

For example, 3rd–order estimates are given by

$$m_j = \frac{2}{5}\iint_{D_A}\left[5 + 24\left(\frac{x_j}{A}\right)^4\right] x_j B_3(\boldsymbol{x}, h)\mathrm{d}^2\boldsymbol{x} + \mathcal{O}\left(\frac{1}{A^3}\right), \qquad j \in \{1, 2\},$$

$$m_3 = \frac{A}{4}\iint_{D_A}\left[5 + 40\left(\frac{x_j}{A}\right)^4 - 128\left(\frac{x_j}{A}\right)^6\right] B_3(\boldsymbol{x}, h)\mathrm{d}^2\boldsymbol{x} + \mathcal{O}\left(\frac{1}{A^3}\right).$$



Recovery of m_1, m_2, m_3 (left to right) for the case of pure (top) and contaminated (bottom) field

Some references

- L. Baratchart, D.P. Hardin, E.A. Lima, E.B. Saff, B.P. Weiss, "Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions", *Inverse Problems 29*, 2013.
- D. Ponomarev, "Magnetisation moment of a bounded 3D sample: asymptotic recovery from planar measurements on a large disk using Fourier analysis", *arXiv:2205.14776*.

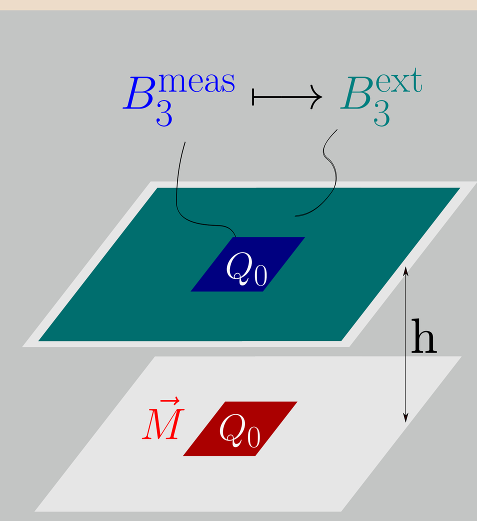
3. Overcoming the data limitation

Experimental limitations

- Measurements are available on a fairly small planar area.
- Only one component of the field $\vec{B}$ is measured: B_3.
- Away from sample, the data is heavily contaminated by noise.

Assumptions

- Planar regular magnetisation:
  $\mathcal{M}(\vec{x}) = \vec{M}(\boldsymbol{x}) \otimes \delta(x_3)$,
  $\operatorname{supp} \vec{M} \subset Q_0 \subset \mathbb{R}^2$,
  $\vec{M} \in \left[W^{1,2}(\mathbb{R}^2)\right]^2 \times L^2(\mathbb{R}^2)$.
- B_3 is measured over Q_0.



Geometry of the extrapolation problem

Extrapolation of measurements

- We mainly aim to extrapolate B_3 on the plane:
  $$B_3(\cdot, h)|_{Q_0} =: B_3^{\text{meas}} \quad \longmapsto \quad B_3^{\text{ext}} := B_3(\cdot, h) \quad \text{on} \quad \mathbb{R}^2.$$
- B_1, B_2 can be reconstructed from B_3^{ext} by suitable bounded integral operators. Moreover, we can obtain $\vec{B}$ on the half–space $\{x_3 > h\}$.
- Extrapolation should at least be stable w.r.t. high–frequency noise.

4. Extrapolation method and results

$$B_3 = K_{12} \star_{Q_0} D_{M_{12}} + K_3 \star_{Q_0} M_3$$

1. Fix J, $N \in \mathbb{N}_+$ sufficiently large.
2. Solve $K_{12} \star_{Q_0} \phi_j = \mu_j \phi_j$ on Q_0.
   $\Rightarrow J$ largest $(\mu_j)_{j=1}^J \subset \mathbb{R}_+$ and $(\phi_j)_{j=1}^J \subset L^2_{\mathbb{R}}(Q_0)$, $\|\phi_j\| = 1$.
3. Compute on $Q_0 \times Q_0$
   $S_J(\mathbf{x}, \mathbf{t}) := \sum_{j=1}^J \frac{\langle K_{12}(\cdot - \mathbf{x}), \phi_j\rangle}{\mu_j}\phi_j(\mathbf{t})$.
4. Solve on Q_0
   $$\begin{cases} K_{12} \star_{Q_0} \varphi_{12}^n + K_3 \star_{Q_0} \varphi_3^n = \lambda_n \varphi_{12}^n, \\ K_{12} \star_{Q_0} \varphi_{12}^n + \mathcal{S}_J K_3 \star_{Q_0} \varphi_3^n = \lambda_n \varphi_3^n. \end{cases}$$
   $\Rightarrow N$ largest in modulus $(\lambda_n)_{n=1}^N \subset \mathbb{R}$ and $\left([\varphi_{12}^n, \varphi_3^n]^T\right)_{n=1}^N \subset \left[L^2_{\mathbb{R}}(Q_0)\right]^2$, $\|\varphi_{12}^n\|^2 + \|\varphi_3^n\|^2 = 1$.
5. Construct $B_3^{\text{ext}}(\mathbf{x}) = \sum_{n=1}^N b_n \widetilde{\varphi}_{12}^n(\mathbf{x})$,
   $b_n := \langle B_3^{\text{meas}}, \varphi_{12}^n\rangle + \langle \mathcal{S}_J B_3^{\text{meas}}, \varphi_3^n\rangle$,
   $\widetilde{\varphi}_{12}^n(\mathbf{x}) := \frac{1}{\lambda_n}\iint_{Q_0}\left[K_{12}(\mathbf{x}-\mathbf{t})\varphi_{12}^n(\mathbf{t}) + K_3(\mathbf{x}-\mathbf{t})\varphi_3^n(\mathbf{t})\right]\mathrm{d}^2\mathbf{t}, \qquad \mathbf{x} \in \mathbb{R}^2$.

Notations

$K_{12}(\mathbf{x}) := -\frac{h}{4\pi\left(|\mathbf{x}|^2 + h^2\right)^{3/2}}$,

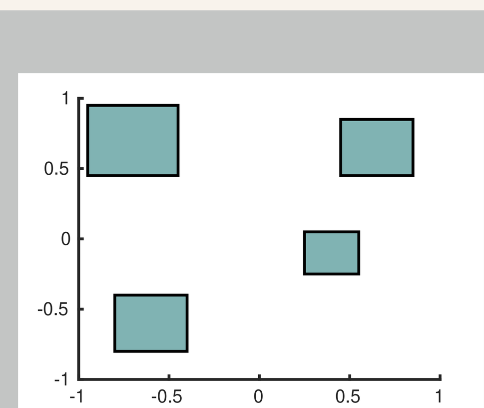
$K_3(\mathbf{x}) := \partial_h K_{12}(\mathbf{x})$,

$D_{M_{12}}(\mathbf{x}) := \partial_1 M_1(\mathbf{x}) + \partial_2 M_2(\mathbf{x})$,

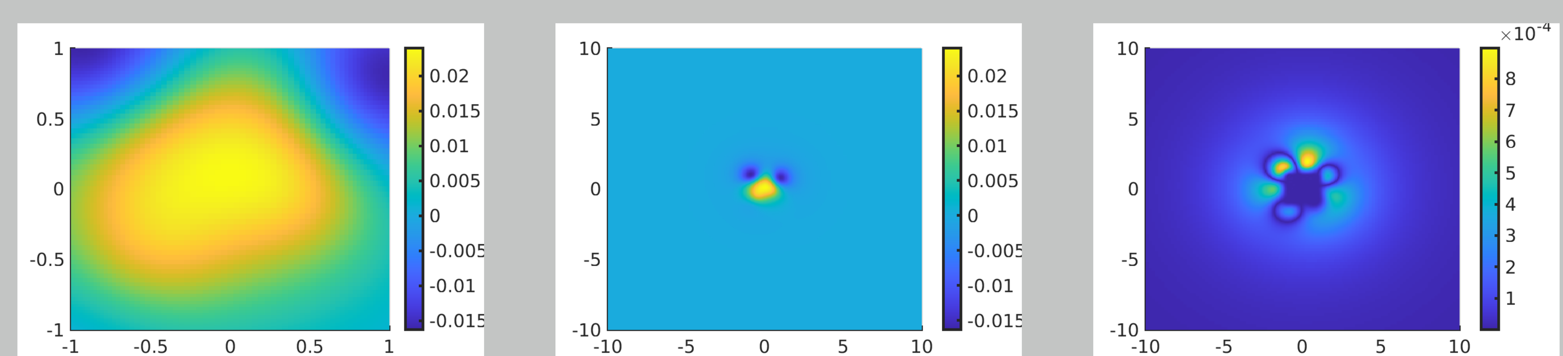
$(K \star_{Q_0} f)(\mathbf{x}) \equiv \iint_{Q_0} K(\mathbf{x}-\mathbf{t}) f(\mathbf{t})\, d^2\mathbf{t}$,

$(\mathcal{S}_J f)(\mathbf{x}) := \iint_{Q_0} S_J(\mathbf{x}, \mathbf{t}) f(\mathbf{t})\, d^2\mathbf{t}$,

$\langle f, g\rangle \equiv \iint_{Q_0} f(\mathbf{t}) g(\mathbf{t})\, d^2\mathbf{t}$.



Support of $\vec{M}$ contained in $Q_0 := [-1, 1]^2$



Original B_3^{meas} (left), extrapolated field B_3^{ext} (middle) and extrapolation error (right) on $[-10, 10]^2$

https://team.inria.fr/factas/

dmitry.ponomarev@inria.fr