Reconstruction of the net magnetisation of a paleomagnetic sample from partial measurements of its magnetic field

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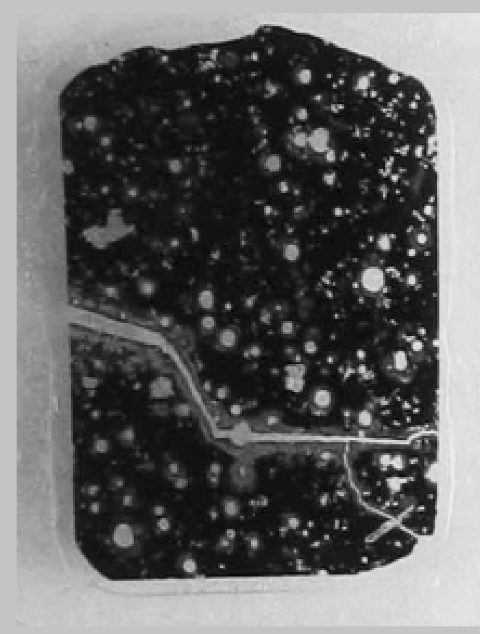
Introduction

SQUID microscope (EAPS lab, MIT)

The process of extraction of relict magnetic information from georocks and meteorites is a challenging task. Due to the weak intensity of the field produced by the remanent magnetisation of a rock, the measurements have to be performed in direct vicinity of the sample and using highly sensitive magnetometric devices such as SQUID and QDM. The basic quantity of interest is the net magnetisation (magnetisation moment vector). Reconstruction of this quantity hinges on effective processing of the experimental data, with the main challenges being the limited measurement area and the noise contamination.



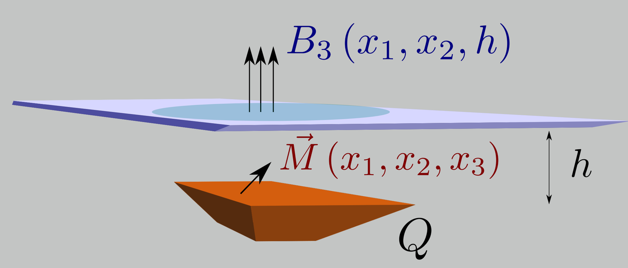
Slice of a magnetised sample (basalt)

1. Setting

Unknown sample magnetisation:

$$\vec{M}(\boldsymbol{x}, x_3) \equiv (M_1, M_2, M_3)(\boldsymbol{x}, x_3),$$

$$\boldsymbol{x} \equiv (x_1, x_2), \quad \vec{x} \equiv (\boldsymbol{x}, x_3).$$



Geometry of the model

Measured component of the magnetic field:

$$B_3(\boldsymbol{x}, h) = \frac{\mu_0}{4\pi} \iiint_Q \frac{3(h - t_3)\left[M_1(\boldsymbol{t}, t_3)(x_1 - t_1) + M_2(\boldsymbol{t}, t_3)(x_2 - t_2)\right] + M_3(\boldsymbol{t}, t_3)\left(2(h - t_3)^2 - |\boldsymbol{x} - \boldsymbol{t}|^2\right)}{\left(|\boldsymbol{x} - \boldsymbol{t}|^2 + (h - t_3)^2\right)^{5/2}} \mathrm{d}^3 t.$$

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$\vec{B} = -\mu_0 \nabla \Phi + \mu_0 \vec{M}$, $\Delta \Phi = \nabla \cdot \vec{M}$, where Φ is the magnetic potential and μ_0 is the magnetic constant.

Desired quantity is the net magnetisation vector:

$$\vec{m} := \iiint_Q \vec{M}(\vec{x})\, \mathrm{d}^3 x \equiv (m_1, m_2, m_3).$$

2. Asymptotic results

When the measurement area is D_A, a disk of sufficiently large radius A, net magnetisation components can be evaluated asymptotically.

- 1st-order estimates:

$$m_j = \frac{2}{\mu_0} \iint_{D_A} x_j B_3(\boldsymbol{x}, h) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A}\right), \qquad j = 1, 2.$$

- 2nd-order estimates:

$$m_j = \frac{2}{\mu_0} \iint_{D_A} \left(1 + \frac{4x_j^2}{3A^2}\right) x_j B_3(\boldsymbol{x}, h) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^2}\right), \qquad j = 1, 2,$$

$$m_3 = \frac{2A}{\mu_0} \iint_{D_A} B_3(\boldsymbol{x}, h) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^2}\right).$$

- 3rd-order estimates:

$$m_j = \frac{2}{5\mu_0} \iint_{D_A} \left[5 + 24\left(\frac{x_j}{A}\right)^4\right] x_j B_3(\boldsymbol{x}, h) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^3}\right), \qquad j = 1, 2,$$

$$m_3 = \frac{A}{4\mu_0} \iint_{D_A} \left[5 + 40\left(\frac{x_j}{A}\right)^4 - 128\left(\frac{x_j}{A}\right)^6\right] B_3(\boldsymbol{x}, h) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^3}\right), \quad j = 1, 2.$$

- 4th-order estimates:

$$m_j = \frac{2}{105\mu_0} \iint_{D_A} \left[105 - 2016\left(\frac{x_j}{A}\right)^4 + 19200\left(\frac{x_j}{A}\right)^6 - 22400\left(\frac{x_j}{A}\right)^8\right] x_j B_3(\boldsymbol{x}, h) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^4}\right), \quad j = 1, 2,$$

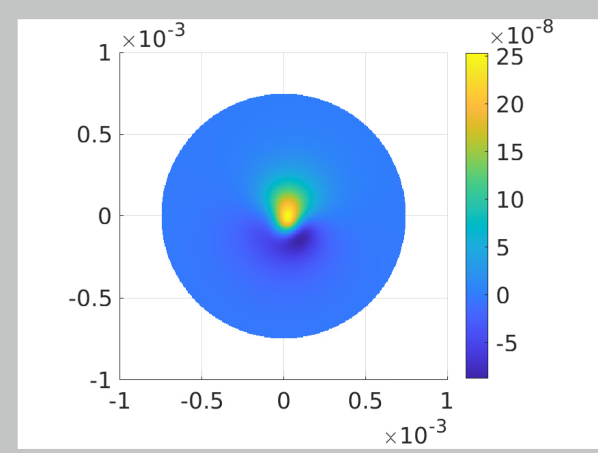
$$m_3 = \frac{A}{24\mu_0} \iint_{D_A} \left[35 + 1792\left(\frac{x_j}{A}\right)^6 - 3200\left(\frac{x_j}{A}\right)^8\right] B_3(\boldsymbol{x}, h) \mathrm{d}^2 x + \mathcal{O}\left(\frac{1}{A^4}\right), \quad j = 1, 2.$$

Some references

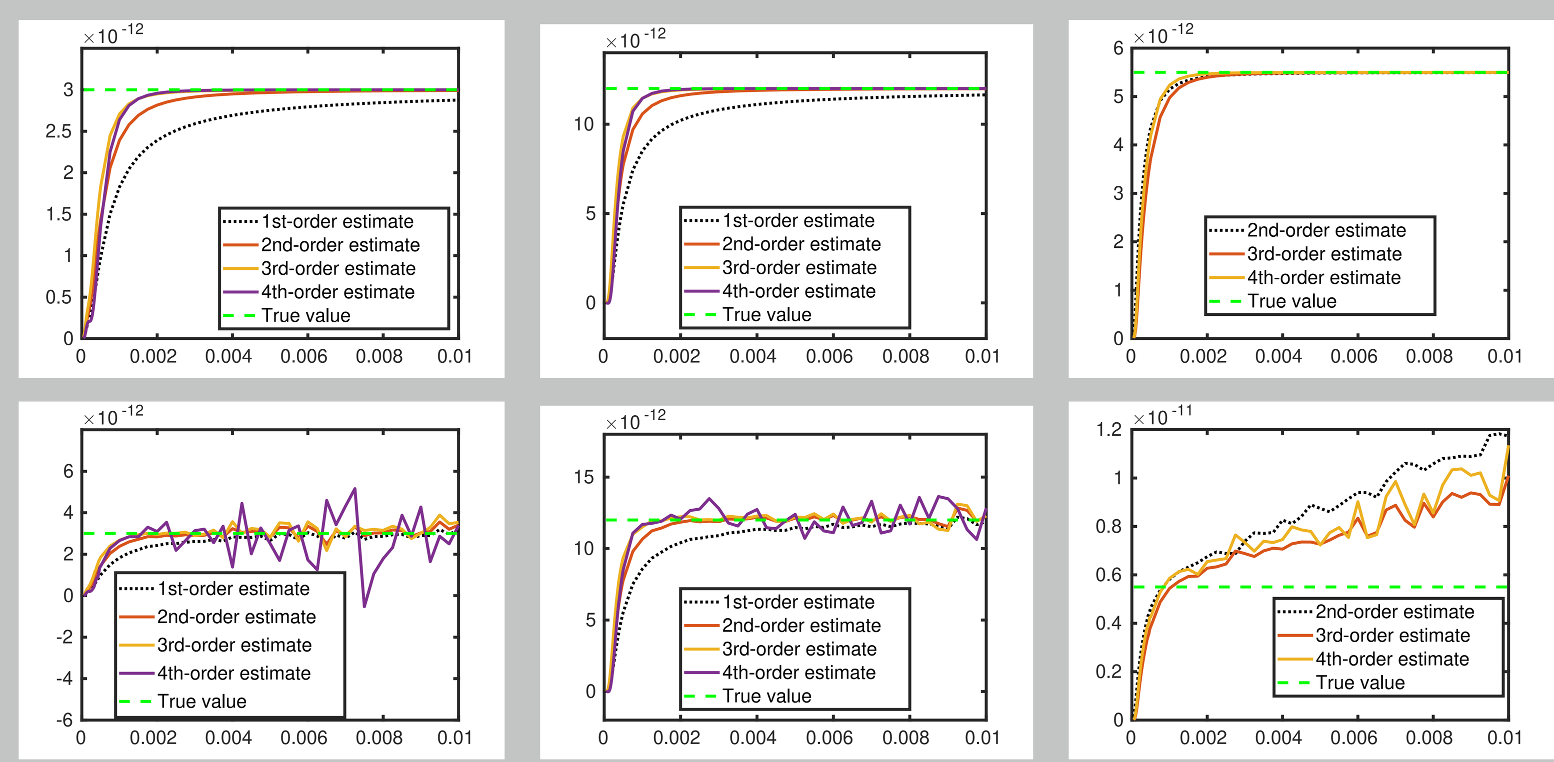
- L. Baratchart, D.P. Hardin, E.A. Lima, E.B. Saff, B.P. Weiss, "Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions", *Inverse Problems 29*, 2013.
- D. Ponomarev, "Magnetisation moment of a bounded 3D sample: asymptotic recovery from planar measurements on a large disk using Fourier analysis", *arXiv:2205.14776*, 2022.

3. Illustration of the asymptotic formulas

Consider the field produced by a synthetic example of 4 magnetic dipoles scattered in the volume $Q = [-0.8, 0.7] \times [-1.1, 1.1] \times [0.2, 2.3] \cdot 5 \cdot 10^{-5}$ m and separated from the $(x_1 x_2)$ measurement plane by height $h = 2.5 \cdot 10^{-5}$ m. Added noise is Gaussian white with SNR $= 20$.



Magnetic field generated by 4 dipoles



Recovery of m_1, m_2, m_3 (left to right) for the case of pure (top) and contaminated (bottom) field

4. Field extrapolation and noise reduction

- Asymptotic formulas may not be directly useful in practice due to a small measurement area.
- Some (higher-order) formulas are quite sensitive to noise.

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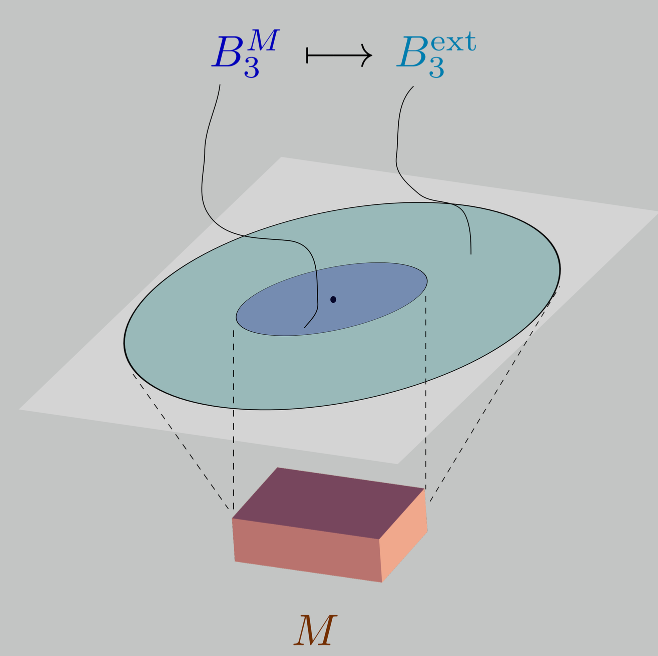
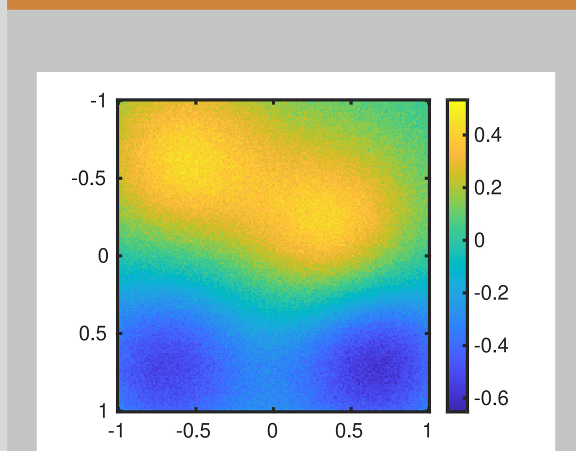
Use appropriate field preprocessing: "spectral extrapolation".

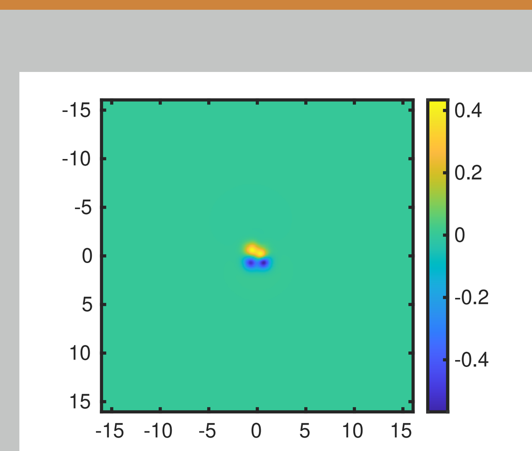


5. Asymptotic formulas with field extrapolation and denoising

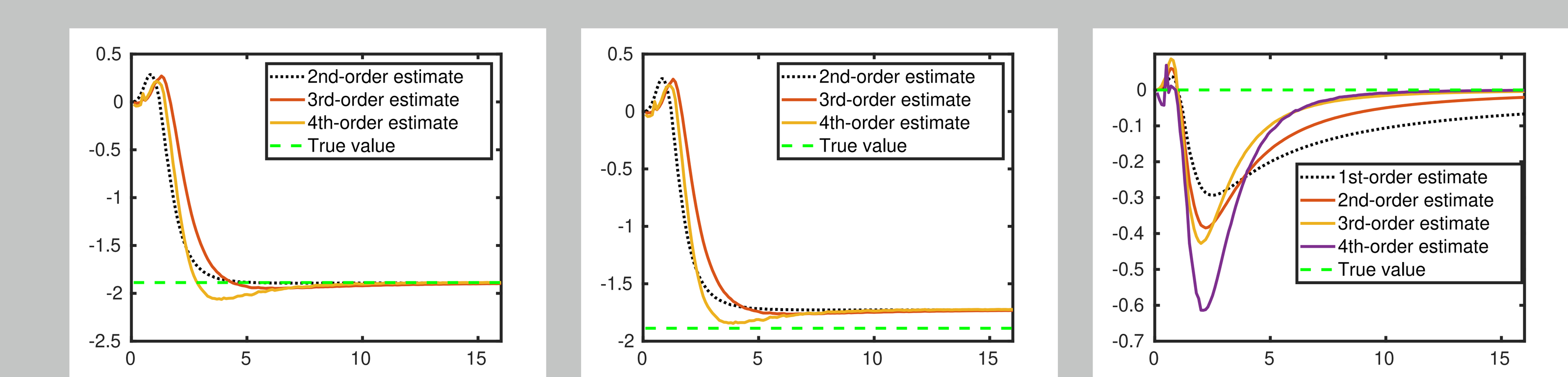


Noisy field of 4 rectangles (B_3^M)

Consider the *non-localised noisy* field produced by 4 planar rectangles, with $M_1 = M_2 \equiv 0$, contained in the square $[-1, 1]^2$ at distance $h = 1$.



Extrapolated field (B_3^{ext})



Recovery of m_3 from pure (left) and 10%-noisy (middle) extrapolated field and validation of $m_1 = 0$ (right).

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