A METHOD TO EXTRAPOLATE THE DATA FOR THE INVERSE MAGNETISATION PROBLEM WITH A PLANAR SAMPLE

DMITRY PONOMAREV[1]

ABSTRACT. A particular instance of the inverse magnetisation problem is considered. It is assumed that the support of a magnetic sample (a source term in the Poisson equation in $\mathbb{R}^3$) is contained in a bounded planar set parallel to the measurement plane. Moreover, only one component of the magnetic field is assumed to be known (measured) over the same planar region in the measurement plane. We propose a method to extrapolate the measurement data to the whole plane relying on the knowledge of the forward operator and the geometry of the problem. The method is based on the spectral decomposition of an auxiliary matrix-function operator. The results are illustrated numerically.

1. INTRODUCTION

Inverse magnetisation problems are inverse source problems for Poisson partial differential equation with the source term being a divergence of some vector field.

We are concerned with a particular instance of such problems which corresponds to a concrete experimental set-up that is used in the Paleomagnetism lab of Earth, Atmospheric & Planetary Sciences Department at MIT (USA) and that can be schematically described as follows. Given measurements of vertical component of the magnetic field B_3 on a patch $Q \subset \mathbb{R}^2$ of the horizontal plane at height $x_3 = h$, one wants to somehow estimate a planar magnetisation distribution $\overrightarrow{M}$ supported inside the region Q of the underlying horizontal plane (i.e., the plane at height $x_3 = 0$). Such a problem is known to be ill-posed [3, 2] for it admits non-unique solutions, and the solutions are not stable with respect to the measured data B_3. To deal with the non-uniqueness issue, additional assumptions (such as unidimensionality) on the magnetisation distribution must be made, see [3, 4]. Alternatively, to avoid such a strong restriction, one can resort to reconstruction of a net magnetisation of a sample (also called magnetisation moment) $\overrightarrow{m} := \iint_Q \overrightarrow{M} d^2\mathbf{x}$, a constant vector that can be recovered uniquely and is of primary physical importance. There have been several approaches proposed to estimate this quantity [1, 5, 6], however, the quality of estimation heavily depends on the size of the region with available data and the presence of noise. This leads us to the problem of data extrapolation and denoising. In the present work, we propose one approach of dealing with this issue which is based on the spectral decomposition of some auxiliary operator that encodes the geometry of the problem and the assumption on support of the magnetisation distribution.

Let us now formulate the problem mathematically. Given an open bounded set $Q \subset \mathbb{R}^2$, we assume that $\overrightarrow{M} \equiv (M_1, M_2, M_3)^T \in \left[W^{1,2} \left(\mathbb{R}^2 \right) \right]^2 \times L^2 \left(\mathbb{R}^2 \right)$, supp $\overrightarrow{M} \subset Q$ and $B_3^{\text{meas}} \in L^2 (Q)$. In the case of non-noisy data, we have $B_3^{\text{meas}} = B_3 |_{Q \times \{h\}}$. The relation between the measured component of the magnetic field B_3 at height $x_3 = h > 0$ and the unknown

[1] Centre Inria d'Université Côte d'Azur, Factas team, 2004 Route des Lucioles, 06902 Biot, France, dmitry.ponomarev@inria.fr.

magnetisation distribution $\overrightarrow{M}$ in the plane $x_3 = 0$ is given by (see, e.g., [2, Sec 3.2] with $\mu_0 = 1$ by choice of units)

$$(1.1) \qquad B_3(\mathbf{x}, h) = -\frac{1}{2} \iint_Q \partial_{x_1} p_h(\mathbf{x} - \mathbf{t}) M_1(\mathbf{t}) \mathrm{d}^2 \mathbf{t} - \frac{1}{2} \iint_Q \partial_{x_2} p_h(\mathbf{x} - \mathbf{t}) M_2(\mathbf{t}) \mathrm{d}^2 \mathbf{t}$$
$$- \frac{1}{2} \iint_Q \partial_h p_h(\mathbf{x} - \mathbf{t}) M_3(\mathbf{t}) \mathrm{d}^2 \mathbf{t}, \qquad \mathbf{x} \in \mathbb{R}^2,$$

where $p_h(\mathbf{x}) := \frac{h}{2\pi(|\mathbf{x}|^2 + h^2)^{3/2}}$ and we adopted the notation $\mathbf{x} \equiv (x_1, x_2)^T$ for two-dimensional vectors.

2. FIELD EXTRAPOLATION

We aim to extrapolate the field B_3^{meas} from Q to the whole $\mathbb{R}^2$. First of all, let us show that this problem makes sense, at least when the measurement data are ideal (i.e., without noise).

Proposition 1. *Given ideal measurement data $B_3^{\mathrm{meas}} = B_3|_{Q \times \{h\}}$ satisfying (1.1) on Q with some functions M_1, M_2, $M_3 \in L^2(\mathbb{R}^2)$, there exists a unique extrapolant B_3^{ext} satisfying (1.1) on $\mathbb{R}^2$ with the same M_1, M_2 and M_3.*

Proof. The existence of the extrapolant B_3^{ext} is immediate from (1.1) due do the fact that each of functions $\partial_{x_1} p_h$, $\partial_{x_2} p_h$ and $\partial_h p_h$ are defined on the whole $\mathbb{R}^2$. Let us show the uniqueness. To this effect, it is useful to rewrite (1.1) in an equivalent form

$$(2.1) \qquad B_3(\mathbf{x}, h) = -\frac{1}{2} \iint_{\mathbb{R}^2} \partial_h p_h(\mathbf{x} - \mathbf{t}) f_M(\mathbf{t}) \mathrm{d}^2 \mathbf{t} = -\frac{1}{2}(\partial_h p_h \star f_M)(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2,$$

with $f_M := \mathscr{R}_1 M_1 + \mathscr{R}_2 M_2 + M_3 \in L^2(\mathbb{R}^2)$, where $\mathscr{R}_1$, $\mathscr{R}_2$ are Riesz transforms, see [3, Thm 2.1], [2, Sec 3.1]. Suppose the contrary, i.e., there is some $L^2(\mathbb{R}^2) \ni \widetilde{f}_M \neq f_M$ such that the corresponding field

$$\widetilde{B}_3(\mathbf{x}, h) := -\frac{1}{2}\left(\partial_h p_h \star \widetilde{f}_M\right)(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2,$$

satisfies $\widetilde{B}_3(\cdot, h) = B_3(\cdot, h) = B_3^{\mathrm{meas}}$ on Q whereas $\widetilde{B_3^{\mathrm{ext}}} := \widetilde{B}_3(\cdot, h) \neq B_3(\cdot, h) =: B_3^{\mathrm{ext}}$ on $\mathbb{R}^2 \backslash \overline{Q}$. It is then immediate that the function $B_3^{\mathrm{diff}} := \widetilde{B}_3(\cdot, h) - B_3(\cdot, h) = -\frac{1}{2}\partial_h p_h \star \left(\widetilde{f}_M - f_M\right)$ vanishes on Q. On the other hand, since B_3^{diff} is real-analytic on $\mathbb{R}^2$ (as the trace of a harmonic function), this function cannot vanish on an open set of $\mathbb{R}^2$ unless it is identically zero on the whole $\mathbb{R}^2$. But $-\frac{1}{2}\partial_h p_h \star$ is a positive operator on $L^2(\mathbb{R}^2)$ (as it corresponds to the Fourier multiplier $\pi|\mathbf{k}|\exp(-2\pi h|\mathbf{k}|)$, $\mathbf{k} \in \mathbb{R}^2$ is the Fourier domain variable). Therefore, we conclude that $\widetilde{f}_M - f_M \equiv 0$ which entails that $\widetilde{B}_3(\cdot, h) = B_3(\cdot, h)$ on $\mathbb{R}^2$ thus contradicting the initial assumption $\widetilde{B_3^{\mathrm{ext}}} \neq B_3^{\mathrm{ext}}$. $\qquad \square$

Despite the positive affirmation of Proposition 1, any set of realistic measurement data is necessarily contaminated by presence of the noise. Hence, in devising an extrapolation method, instability with respect to small perturbations of the data should be also kept in mind. When using spectral approaches, stabilisation can be achieved by means of low-pass filtering as typical noise has a much larger high-frequency content compared to the exact field data. We shall employ this idea here by putting a spectral approach at the heart of the present extrapolation method.

First, we are going to describe the method from the viewpoint of practical implementation, and then outline its justification. For convenience, let us set:

$$K_{12} := -\frac{1}{2} p_h, \qquad K_3 := -\frac{1}{2} \partial_h p_h, \qquad (K \star_Q f)(\mathbf{x}) \equiv \iint_Q K(\mathbf{x} - \mathbf{t}) f(\mathbf{t}) \, d^2 \mathbf{t},$$

$$\langle f, g \rangle \equiv \iint_Q f(\mathbf{t}) g(\mathbf{t}) \, d^2 \mathbf{t}, \qquad \|f\| \equiv \|f\|_{L^2(Q)}.$$

Algorithm.

(i) Fix positive numbers $J, N \in \mathbb{N}_+$ sufficiently large (for ideal data B_3^{meas}).

(ii) Find J largest in modulus eigenvalues $|\mu_1| \geq |\mu_2| \geq \ldots \geq |\mu_J| > 0$ and corresponding real-valued eigenfunctions $(\phi_j)_{j=1}^J \subset L^2(Q)$ satisfying $K_{12} \star_Q \phi_j = \mu_j \phi_j$ on Q and $\|\phi_j\| = 1$ for $j = 1, \ldots, J$.

(iii) For any $f \in L^2(Q)$, define the operator $\mathscr{S}_J : L^2(Q) \to L^2(Q)$ as

(2.2) $$(\mathscr{S}_J f)(\mathbf{x}) := \iint_Q S_J(\mathbf{x}, \mathbf{t}) f(\mathbf{t}) \, d^2 \mathbf{t}, \qquad S_J(\mathbf{x}, \mathbf{t}) := -\sum_{j=1}^J \frac{\langle \partial_h p_h(\cdot - \mathbf{x}), \phi_j \rangle}{2\mu_j} \phi_j(\mathbf{t}).$$

(iv) Find N largest eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ and corresponding real-valued eigenvectors $\left([\varphi_{12}^n, \varphi_3^n]^T \right)_{n=1}^N \subset [L^2(Q)]^2$ satisfying

(2.3) $$\begin{cases} K_{12} \star_Q \varphi_{12}^n + K_3 \star_Q \varphi_3^n = \lambda_n \varphi_{12}^n \\ K_{12} \star_Q \varphi_{12}^n + \mathscr{S}_J K_3 \star_Q \varphi_3^n = \lambda_n \varphi_3^n \end{cases} \quad \text{on } Q$$

and $\|\varphi_{12}^n\|^2 + \|\varphi_3^n\|^2 = 1$ for $n = 1, \ldots, N$.

(v) Given $B_3^{\text{meas}} \in L^2(Q)$, construct the field extrapolant as

(2.4) $$B_3^{\text{ext}}(\mathbf{x}) = \sum_{n=1}^N b_n \widetilde{\varphi}_{12}^n(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2,$$

where, for $n = 1, \ldots, N$,

(2.5) $$b_n := \langle B_3^{\text{meas}}, \varphi_{12}^n \rangle + \langle \mathscr{S}_J B_3^{\text{meas}}, \varphi_3^n \rangle,$$

(2.6) $$\widetilde{\varphi}_{12}^n(\mathbf{x}) := \frac{1}{\lambda_n} \iint_Q [K_{12}(\mathbf{x} - \mathbf{t}) \varphi_{12}^n(\mathbf{t}) + K_3(\mathbf{x} - \mathbf{t}) \varphi_3^n(\mathbf{t})] \, d^2 \mathbf{t}, \qquad \mathbf{x} \in \mathbb{R}^2.$$

Towards justification of the method. Let us start by rewriting (1.1) in a slightly different form employing integration by parts in the first 2 terms:

(2.7) $$B_3(\mathbf{x}, h) = (K_{12} \star_Q D_{M_{12}})(\mathbf{x}) + (K_3 \star_Q M_3)(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2,$$

where $D_{M_{12}}(\mathbf{t}) := \partial_{t_1} M_1(\mathbf{t}) + \partial_{t_2} M_2(\mathbf{t}) \in L^2(Q)$. In particular, on Q, we have $B_3^{\text{meas}}(\mathbf{x}) = (K_{12} \star_Q D_{M_{12}})(\mathbf{x}) + (K_3 \star_Q M_3)(\mathbf{x})$ and applying to this equation the operator $\mathscr{S}_J$ defined in (2.2), we can reformulate both equations as

(2.8) $$\begin{bmatrix} K_{12} \star_Q & K_3 \star_Q \\ \mathscr{S}_J K_{12} \star_Q & \mathscr{S}_J K_3 \star_Q \end{bmatrix} \begin{bmatrix} D_{M_{12}} \\ M_3 \end{bmatrix} = \begin{bmatrix} B_3^{\text{meas}} \\ \mathscr{S}_J B_3^{\text{meas}} \end{bmatrix} \qquad \text{on } Q.$$

Since the truncated Poisson operator $p_h \star_Q$ is positive and compact on $L^2(Q)$, its range is dense in $L^2(Q)$. We hence have

$$\iint_Q K_{12}(\mathbf{t} - \mathbf{y}) S_J(\mathbf{x}, \mathbf{y}) \, d^2 \mathbf{y} = K_3(\mathbf{t} - \mathbf{x}) + r_J(\mathbf{t}, \mathbf{x}), \qquad \mathbf{x}, \mathbf{t} \in Q,$$

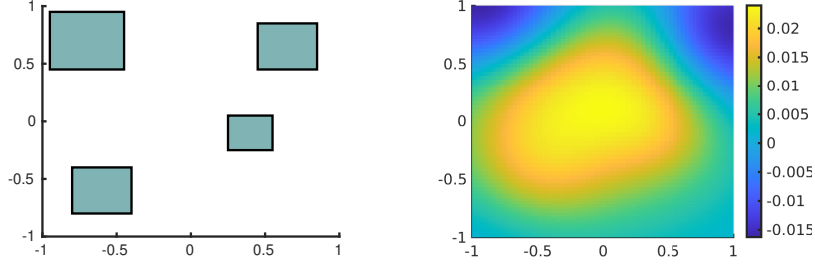FIGURE 3.1. Planar magnetisation support (left) and measured data B_3^{meas} (right) on $[-1,1]^2$

$$r_J(\mathbf{t}, \mathbf{x}) := \sum_{j=1}^{J} \left\langle K_3(\cdot - \mathbf{x}), \phi_j \right\rangle \phi_j(\mathbf{t}) - K_3(\mathbf{t} - \mathbf{x}), \qquad \mathbf{x}, \mathbf{t} \in Q,$$

with $\|r_J(\cdot, \mathbf{x})\| \to 0$ as $J \to \infty$ uniformly for any $\mathbf{x} \in Q$. Moreover, from the definition of S_J in (2.2), it is easy to verify that

$$\iint_Q S_J(\mathbf{x}, \mathbf{y}) K_3(\mathbf{y} - \mathbf{t}) d^2\mathbf{y} = \iint_Q K_3(\mathbf{x} - \mathbf{y}) S_J(\mathbf{t}, \mathbf{y}) d^2\mathbf{y}, \qquad \mathbf{x}, \mathbf{t} \in Q.$$

Consequently, the matrix-operator $\mathscr{L}_J : \left[L^2(Q)\right]^2 \to \left[L^2(Q)\right]^2$ defined as

$$\mathscr{L}_J := \begin{bmatrix} K_{12}\star_Q & K_3\star_Q \\ K_3\star_Q & \mathscr{S}_J K_3\star_Q \end{bmatrix}$$

is self-adjoint, and (2.8) rewrites as

$$(2.9) \qquad \mathscr{L}_J \begin{bmatrix} D_{M_{12}} \\ M_3 \end{bmatrix} = \begin{bmatrix} B_3^{\text{meas}} \\ \mathscr{S}_J B_3^{\text{meas}} \end{bmatrix} - \begin{bmatrix} 0 \\ \iint_Q r_J(\mathbf{t}, \cdot) D_{M_{12}}(\mathbf{t}) d^2\mathbf{t} \end{bmatrix} \qquad \text{on } Q,$$

where the second term on the right-hand side can be made arbitrary small for sufficiently large J. Since the operator $\mathscr{L}_J$ is also compact, the spectral theorem applies and motivates the algorithm above. Further details will be published elsewhere.

3. NUMERICAL ILLUSTRATION

To test the method numerically, we consider the field B_3^{meas} on $Q = [-1,1]^2$ which is produced by a magnetisation whose planar support consists of 4 rectangles contained in Q, see Figure 3.1. Choosing $N = J = 80$ in the described above algorithm, we obtain an extrapolant B_3^{ext}. In Figure 3.2, B_3^{ext} is shown on a rectangle $[-10, 10]^2$, together with a pointwise error. The relative L^2–error in this region is around 7%.

4. CONCLUSIONS

Motivated by improvement of various aspects of inverse magnetisation problem, we have considered an issue of the field extrapolation. After proving the uniqueness of the extrapolant (for the ideal data), we have proposed an approach which reduces the problem by means of solving an auxiliary equation. The method is constructed based on spectral decompositions of the operators of the reduced and the auxiliary problems and hence can be referred to as a "double-spectral" approach. Further analysis and details on the choice of spectral truncation parameters J and N featuring in the presented algorithm will be discussed in a separate paper.
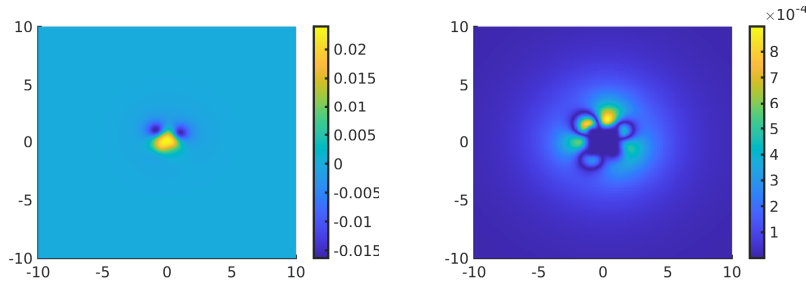
FIGURE 3.2. Extrapolated field B_3^{ext} (left) and extrapolation error (right) on $[-10, 10]^2$

REFERENCES

[1] Baratchart, L., Chevillard, S., Hardin, D., Leblond, J., Lima, E., Marmorat, J. P.: Magnetic moment estimation and bounded extremal problems. Inverse Problems and Imaging **13** (1), 39–67 (2019).

[2] Baratchart, L., Chevillard, S., Leblond, J. Silent and equivalent magnetic distributions on thin plates. Harmonic Analysis, Function Theory, Operator Theory, and their Applications **22**, 17pp. (2018)

[3] Baratchart, L., Hardin, D. P., Lima, E. A., Saff, E. B., Weiss, B. P.: Characterizing kernels of operators related to thin-plate magnetizations via generalizations of Hodge decompositions. Inverse Problems **29** (1), 29pp (2013).

[4] Lima, E. A., Weiss, B. P., Baratchart, L., Hardin, D. P., Saff, E. B.: Fast inversion of magnetic field maps of unidirectional planar geological magnetization. J. Geophys. Res.: Solid Earth **118** (6), 2723–2752 (2013).

[5] Lima, E. A., Weiss, B. P., Borlina, C. S., Baratchart, L., Hardin, D. P.: Estimating the net magnetic moment of geological samples from planar field maps using multipoles. Geochemistry, Geophysics, Geosystems **24**, 38pp (2023).

[6] Ponomarev, D.: Magnetisation moment of a bounded 3D sample: asymptotic recovery from planar measurements on a large disk using Fourier analysis. arXiv:2205.14776, 31pp (2022).