Research Article

Juliette Leblond and Dmitry Ponomarev*

Recovery of harmonic functions from partial boundary data respecting internal pointwise values

DOI: 10.1515/jiip-2015-0089
Received September 28, 2015; revised February 15, 2016; accepted February 15, 2016

Abstract: We consider partially overdetermined boundary-value problem for Laplace PDE in a planar simply connected domain with Lipschitz boundary $\partial\Omega$. Assuming Dirichlet and Neumann data available on $\Gamma \subset \partial\Omega$ to be real-valued functions in $W^{1/2,2}(\Gamma)$ and $L^2(\Gamma)$ classes, respectively, we develop a non-iterative method for solving this ill-posed Cauchy problem choosing L^2 bound of the solution on $\partial\Omega \setminus \Gamma$ as a regularizing parameter. The present complex-analytic approach also naturally allows imposing additional pointwise constraints on the solution which, on practical side, can help incorporating outlying boundary measurements without changing the boundary into a less regular one.

Keywords: Partially overdetermined boundary value problems, Laplace PDE, best constrained approximation, interpolation in Hardy spaces, ill-posed inverse problems, regularization techniques, representation of analytic functions from partial boundary data

MSC 2010: 35N25, 35Q99, 30E10, 30E25, 31A25, 35J25, 41A65

1 Introduction

Many stationary physical problems are formulated in terms of reconstruction of a harmonic function in a planar domain from partially available measurements on its boundary. As it is often the case, the values of both the function and its normal derivative are available only on part of the boundary whereas the main interest is to determine the values inside the domain or on the unaccessible part of the boundary, or sometimes even the position of this complementary part of the boundary [2]. The planar formulation is a simplification that typically arises from original three-dimensional settings whose symmetry properties allow reformulation of the model in dimension two.

The Cauchy problem for Laplace equation is known to be ill-posed: the famous Hadamard's example demonstrates the lack of continuous dependence of the solution on boundary data. This reveals the necessary compatibility between Dirichlet and Neumann data for the existence of physically meaningful solution and advocates use of regularization techniques.

Partially overdetermined problems for the elliptic operators have been considered vastly in various frameworks (see [19] and references therein) and different methods of their regularization and solution have been developed and investigated.

In the present work, we revisit the very classical setting – Laplace PDE on a simply connected domain with Lipschitz boundary. Namely, we consider the prototypical case where the domain is the unit disk $\Omega = \mathbb{D}$,

Juliette Leblond: Centre de recherche INRIA Sophia-Antipolis Méditerranée, 2004, route des Lucioles — BP 93, 06 902 Sophia Antipolis Cedex, France, e-mail: juliette.leblond@inria.fr
***Corresponding author: Dmitry Ponomarev:** Centre de recherche INRIA Sophia-Antipolis Méditerranée, 2004, route des Lucioles — BP 93, 06 902 Sophia Antipolis Cedex, France, e-mail: dmvpon@gmail.com

which is justified by the conformal invariance of Laplace operator. We assume real-valuedness and appropriate regularity of the boundary data on a strict subset $\Gamma \subset \mathbb{T}$ required for the existence of a unique weak $W^{1,2}(\Omega)$ solution:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = u_0, \quad \partial_n u = w_0 & \text{on } \Gamma \text{ with } u_0 \in W^{1/2,2}(\Gamma),\ w_0 \in L^2(\Gamma). \end{cases} \tag{1.1}$$

We employ complex-analytic approach which has proven to be rather efficient in dealing with this [3, 6, 10, 11] and more general formulations of the problem: annular setting [21, 24], conductivity PDE [16] and their mixture [4].

Recall that if a function $g = u + iv$ is analytic (holomorphic), then u and v are real-valued harmonic functions satisfying the Cauchy–Riemann equations $\partial_n u = \partial_t v$, $\partial_t u = -\partial_n v$, where the partial derivatives are taken with respect to polar coordinates. Applied to problem (1.1), the first of these equations suggests that knowing w_0, one can, up to an additive constant, recover v on Γ, and therefore both u_0 and w_0 define the trace on Γ of the function g analytic inside Ω. However, the knowledge of an analytic function on a subset $\Gamma \subset \mathbb{T}$ of positive measure completely defines this function inside the whole domain (unit disk $\mathbb{D}$) [18, 26]. Of course, available data u_0, w_0 on Γ may not be compatible to yield the restriction of an analytic function onto Γ. This fact illustrates ill-posedness of the problem from the complex analysis point of view. At the same time, it leads to a natural regularization scheme that consists of finding a compatible set of data which is the closest to the original one and whose continuation behaves well on the unaccessible part of the boundary.

The described procedure can be formalized as a best norm-constrained approximation problem in Hardy space for the disk casted in the works [3, 6]. Pursuing this approach, we extend previously obtained results as follows.

First of all, we generalize the method in order to allow internal pointwise constraints on the solution. We rederive solution formula and carry out analysis of the approximation quality for this case. One practical aspect of this modification might be a possibility to effectively process measurements from sensors positioned off the naturally smooth boundary by clustering these outlying measurements into a few points located inside the domain. We note that here internal pointwise data do make sense due to the analytical structure of the present framework – an advantage of working in Hardy rather than Lebesgue spaces. The possibility of imposing finite or infinite number of internal pointwise constraints on analytic function in the disk is classical [27] and has been studied from different viewpoints (e.g. [9]).

Second, we improve the previous solution algorithm which was an iterative procedure. As before, the solution formula is implicit for it contains a parameter to be chosen to satisfy the regularization constraint. However, if this adjustment previously had to be done by dichotomy, we now provide an expression allowing one to estimate this parameter directly from the regularization bound and thus avoid repetitive solution of the problem.

Lastly, we prove stability of the regularized problem with respect to all input data – a technical issue that appears not to have been raised before.

The paper is organized in the following way. Section 2 provides an introduction to the theory of Hardy spaces which are essential functional spaces in the present approach. In Section 3, we formulate the problem in that framework, prove existence of a unique solution, obtain its implicit characterization and additionally discuss the choice of some interpolation function used in order to prescribe desired values inside the domain. In Section 4, we obtain specific balance relations governing the approximation rate on a given subset of the circle and the solution growth on its complement. This sheds light on the quality of the approximate solution depending on a choice of some auxiliary parameters. We finally introduce a novel series expansion method for evaluation of certain characteristics governing the solution quality. Combined with a previously obtained implicit characterization formula for the minimizer (best approximant), it yields a practical way of solving the bounded extremal problem. We further look into sensitivity of the solution to perturbations of all input data in Section 5 raising the stability issue and providing technical estimates. We finalize the work with Section 6 by presenting numerical illustrations of the method, a short discussion about the choice of technical parameters and a suggestion of a possible computational strategy. Some concluding remarks are given in Section 7.

2 Essential background on Hardy spaces

Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk with boundary $\mathbb{T}$.

Hardy spaces $H^2(\mathbb{D})$, $H^\infty(\mathbb{D})$ can be defined as classes of holomorphic functions on the disk with finite norms

$$\|F\|_{H^2} = \sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|F(re^{i\theta})|^2\,d\theta\right)^{1/2}, \quad \|F\|_{H^\infty} = \sup_{|z|<1}|F(z)|.$$

These are Banach spaces that enjoy plenty of remarkable properties that have been studied in detail over the years [15, 17, 20, 27]. It is crucial that functions in Hardy spaces have boundary values defined pointwise almost everywhere on $\mathbb{T}$ with the limit existing along any non-tangential path from inside the disk. Moreover, H^2 is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\,d\theta,$$

and this space can be characterized as a subspace of $L^2(\mathbb{T})$-functions whose Fourier coefficients of negative index vanish. In fact, we have the decomposition $L^2(\mathbb{T}) = H^2 \oplus \bar{H}_0^2$, where the orthogonal complement $\bar{H}_0^2$ is a space consisting of $L^2(\mathbb{T})$-functions which may only possess Fourier coefficients of strictly negative index, that is, functions holomorphic in the exterior disk $\mathbb{C} \setminus \bar{\mathbb{D}}$.

Another important property of Hardy classes is the possibility to perform certain structural factorizations, see for instance [27]. In particular, we can factor out the zeros of a Hardy function without reducing its norm: if $f \in H^2$ and $f(z_j) = 0$, $z_j \in \mathbb{D}$, $j = 1, \dots, N$, then $f = bg$ with zero-free $g \in H^2$ and the finite Blaschke product $b \in H^\infty$, $\|b\|_{H^\infty} = 1$ defined as

$$b(z) = e^{i\phi_0}\prod_{j=1}^{N}\left(\frac{z - z_j}{1 - \bar{z}_j z}\right) \tag{2.1}$$

for some constant $\phi_0 \in [0, 2\pi]$.

Any function in H^2, being analytic and sufficiently regular on $\mathbb{T}$, admits an integral representation in terms of its boundary values and is thus uniquely determined by means of the Cauchy formula. However, it is also possible to recover a function f holomorphic in $\mathbb{D}$ from its values on a subset of the boundary $I \subset \mathbb{T}$ using the so-called Carleman's formulas [1, 18]. Write $\mathbb{T} = I \cup J$, where I and J are measurable sets with the Lebesgue measures $|I|$ and $|J|$, respectively.

Proposition 2.1. *Assume $|I| > 0$ and let $\Phi \in H^\infty$ be any function such that $|\Phi| > 1$ in $\mathbb{D}$ and $|\Phi| = 1$ on J. Then, $f \in H^2$ can be represented from $f|_I$ as*

$$f(z) = \frac{1}{2\pi i}\lim_{\alpha\to\infty}\int_I \frac{f(\xi)}{\xi - z}\left[\frac{\Phi(\xi)}{\Phi(z)}\right]^\alpha d\xi, \tag{2.2}$$

where the convergence is uniform on compact subsets of $\mathbb{D}$.

Remark 2.1. Using the isometry $H^2 \to \bar{H}_0^2$, $f(z) \mapsto \frac{1}{z}\overline{f(\frac{1}{\bar{z}})}$, $z \in \mathbb{D}$, we check that Proposition 2.1 also applies to functions in $\bar{H}_0^2$. In particular, for $f \in H^2$ or $\bar{H}_0^2$, $|I| > 0$, we have the implication $f|_I = 0 \implies f \equiv 0$ in $\mathbb{D}$ or $\mathbb{C} \setminus \bar{\mathbb{D}}$, respectively.

An operator A is called a Toeplitz operator on H^2 if its matrix in the Fourier basis has constant elements along all diagonals: $A_{k,m} := \langle Az^k, z^m\rangle_{L^2(\mathbb{T})}$ depends only on $(k - m)$ for $k, m = 0, 1, 2, \dots$.

The following spectral result for Toeplitz operators is known as Hartman–Wintner theorem. Its proof can be found in [14, 25] and, in self-consistent manner, in [8].

Proposition 2.2. *Let $\xi \in L^\infty(\mathbb{T})$, $\mathbb{T} \to \mathbb{R}$, be a symbol defining the Toeplitz operator*

$$T_\xi : H^2 \to H^2, \quad F \mapsto T_\xi(F) = P_+(\xi F).$$

Then, the operator spectrum is $\sigma(T_\xi) = [\operatorname{ess\,inf}\xi, \operatorname{ess\,sup}\xi] \subset \mathbb{R}$.

Define the Toeplitz operator ϕ with symbol χ_J, representing the indicator function of the set J, as

$$H^2 \to H^2, \quad F \mapsto \phi(F) = P_+(\chi_J F), \tag{2.3}$$

where we let P_+ denote the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. Similarly, $P_- := I - P_+$ defines the orthogonal projection onto $\bar{H}_0^2$.

Proposition 2.3. *The Toeplitz operator ϕ is an injection on H^2. Moreover, this result is valid if the operator domain is extended to the whole $L^2(\mathbb{T})$.*

Proof. By the orthogonal decomposition $L^2 = H^2 \oplus \bar{H}_0^2$, we have $\chi_J g = P_+(\chi_J g) + P_-(\chi_J g)$. Now, if $P_+(\chi_J g) = 0$, then $\chi_J g$ is an $\bar{H}_0^2$-function vanishing on I and hence, by Remark 2.1, must be identically zero. □

We finally formulate the density results which will be essential for stating the main problem.

Proposition 2.4. *Let $J \subset \mathbb{T}$ be a subset of non-full measure, that is, $|I| = |\mathbb{T} \setminus J| > 0$. Then, the restriction $H^2|_J := (\operatorname{tr} H^2)|_J$ is dense in $L^2(J)$.*

Proof. We argue by contradiction: assume that there is a non-zero $f \in L^2(J)$ orthogonal to $H^2|_J$. Then, extending it by zero on I, we denote the extended function as $\tilde{f}$. We thus have $\langle \tilde{f}, g \rangle_{L^2(\mathbb{T})} = 0$ for all $g \in H^2$ which implies $\tilde{f} \in \bar{H}_0^2$ and hence, by Remark 2.1, $f \equiv 0$. □

Proposition 2.5. *Assume $|I| > 0$, $f \in L^2(I)$. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of H^2-functions such that*

$$\lim_{n \to \infty} \|f - g_n\|_{L^2(I)} = 0.$$

Then, $\|g_n\|_{L^2(J)} \to \infty$ as $n \to \infty$ unless f is the trace of an H^2-function.

Proof. Assume that f is not the trace on I of some H^2-function, but at the same time $\lim_{n\to\infty} \|g_n\|_{L^2(J)} < \infty$. Then, by hypothesis, the sequence $\{g_n\}_{n=1}^{\infty}$ is bounded not only in $L^2(J)$ but also in H^2. Since H^2 is reflexive, by the Banach–Alaoglu theorem [23], the closed unit ball in H^2 is weakly compact, therefore we can extract a subsequence $\{g_{n_k}\}$ that converges weakly in H^2 to some $g \in H^2$. However, since $g_n \to f$ in $L^2(I)$, we must have $f = g|_I$ yielding a contradiction. □

3 Approximation problem with pointwise data

3.1 Formulation of the problem

Motivated by the Cauchy problem (1.1), we are aiming to find an analytic function which is consistent with available physical measurements. Proposition 2.4 allows a possibility for finding an analytic function with arbitrary close correspondence to given data, but on the other hand, Proposition 2.5 implies that any approximation scheme would be unstable with respect to input data which are unevitably contaminated by noise, round-off errors, etc. To make the problem well-posed, constraint has to be added and hence this leads one to search for a constrained best approximant of an L^2-function defined on a subset of the boundary by the trace of an H^2-function. Such problems have already been investigated extensively in the half-plane setting [22] and on the disk [3, 6, 7].

In the present work, we extend the latter results, namely, we consider the problem of finding an H^2-function which takes prescribed values $\{\omega_j\}_{j=1}^N \in \mathbb{C}$ at interior points $\{z_j\}_{j=1}^N \in \mathbb{D}$ and which best approximates a given $L^2(I)$-function on a subset of the boundary $I \subset \mathbb{T}$ while remaining close enough to another $L^2(J)$-function on the complementary part $J = \mathbb{T} \setminus I$. Note that $I = \Gamma$ in the notation of Section 1.

We proceed with a technical formulation of this problem. Assuming given interpolation values at distinct interior points $\{z_j\}_{j=1}^N \in \mathbb{D}$, we let $\psi \in H^2$ be some fixed function satisfying the pointwise interpolation conditions

$$\psi(z_j) = \omega_j \in \mathbb{C}, \quad j = 1, \ldots, N. \tag{3.1}$$

Then, any interpolating function in H^2 fulfilling these conditions can be written as $\tilde{g} = \psi + bg$ for arbitrary $g \in H^2$ with $b \in H^\infty$ the finite Blaschke product defined in (2.1).

Let us assume $|I|, |J| > 0$ and write $f = f|_I \vee f|_J$ to mean a function defined on the whole $\mathbb{T}$ through its values given on I and J.

For $h \in L^2(J)$, $M \geq 0$, let us introduce the following functional spaces:

$$\begin{aligned} \mathcal{A}^{\psi,b} &:= \{\tilde{g} \in H^2 : \tilde{g} = \psi + bg,\ g \in H^2\}, \\ \mathcal{B}^{\psi,b}_{M,h} &:= \{g \in H^2 : \|\psi + bg - h\|_{L^2(J)} \leq M\}, \\ \mathcal{C}^{\psi,b}_{M,h} &:= \{f \in L^2(I) : f = \psi|_I + bg|_I,\ g \in \mathcal{B}^{\psi,b}_{M,h}\}. \end{aligned} \tag{3.2}$$

We then have the inclusions

$$\mathcal{C}^{\psi,b}_{M,h} \subseteq \mathcal{A}^{\psi,b}|_I \subseteq H^2|_I \subset L^2(I),$$

and $\mathcal{C}^{\psi,b}_{M,h} = (\psi + b\mathcal{B}^{\psi,b}_{M,h})|_I \neq \emptyset$ since, in general (for arbitrary $h \in L^2(J)$), $\mathcal{B}^{\psi,b}_{M,h} \neq \emptyset$ for $M > 0$ as follows from Proposition 2.4.

In this set-up, a solution to the approximation problem is

$$\tilde{g}_0 := \psi + bg_0 \in \mathcal{A}^{\psi,b} \quad \text{such that} \quad g_0 = \arg\min_{g \in \mathcal{B}^{\psi,b}_{M,h}} \|\psi + bg - f\|_{L^2(I)}, \tag{3.3}$$

i.e. a best H^2-approximant to f on I which fulfils the interpolation conditions (3.1) and does not deviate much from the reference h on J: $\|\tilde{g}_0 - h\|_{L^2(J)} \leq M$. In view of Proposition 2.5, the L^2-constraint on J is crucial for the problem to be well-posed whenever $f \notin \mathcal{A}^{\psi,b}|_I$ (which is always the case in context of practical applications). In other words, we assume that

$$g|_I \neq \bar{b}(f - \psi), \tag{3.4}$$

i.e. there is no $\tilde{g} = \psi + bg \in H^2$ whose trace on I is exactly the given function $f \in L^2(I)$, and at the same time remains within the L^2-distance M from h on J. This motivates the choice (3.2) for the space of admissible solutions $\mathcal{B}^{\psi,b}_{M,h}$.

3.2 Bounded extremal problem

Given $f \in L^2(I)$, solving the analytic function approximation problem with pointwise constraints is tantamount to finding a solution to the following bounded extremal problem:

$$\min_{g \in \mathcal{B}^{\psi,b}_{M,h}} \|\psi + bg - f\|_{L^2(I)}. \tag{3.5}$$

Existence and uniqueness of a solution to (3.5) can be reduced to what has been proved in a general setting in [6]. Here we present a slightly different proof taking advantage of the Hilbertian setting.

Theorem 3.1. *For any $f \in L^2(I)$, $h \in L^2(J)$, $\psi \in H^2$, $M \geq 0$ and $b \in H^\infty$ defined as* (2.1), *there exists a unique solution to the bounded extremal problem* (3.5).

Proof. By the existence of a best approximation projection onto a non-empty closed convex subset of a Hilbert space [13], it is required to show that the space of restrictions $\mathcal{B}^{\psi,b}_{M,h}|_I$ is a closed convex subset of $L^2(I)$. Convexity is a direct consequence of the triangle inequality:

$$\|\alpha(bg_1 + \psi - h) + (1 - \alpha)(bg_2 + \psi - h)\|_{L^2(J)} \leq \alpha M + (1 - \alpha)M = M$$

for any $g_1, g_2 \in \mathcal{B}^{\psi,b}_{M,h}$ and $\alpha \in [0, 1]$.

We will now show the closedness property. Let $\{g_n\}_{n=1}^\infty$ be a sequence of $\mathcal{B}^{\psi,b}_{M,h}$-functions which converges in $L^2(I)$ to some function g: $\|g - g_n\|_{L^2(I)} \to 0$ as $n \to \infty$. We need to prove that $g \in \mathcal{B}^{\psi,b}_{M,h}$.

We note that $g \in H^2|_I$, since otherwise, by Proposition 2.5, $\|g_n\|_{L^2(J)} \to \infty$ as $n \to \infty$, which would contradict the fact that $g_n \in \mathcal{B}^{\psi,b}_{M,h}$ starting with some n. Therefore, $\psi + bg \in H^2$ and $\langle \psi + bg, \xi \rangle_{L^2(\mathbb{T})} = 0$ for any $\xi \in \bar{H}^2_0$, which implies that

$$\langle \psi + bg, \xi \rangle_{L^2(I)} = \langle (\psi + bg) \vee 0, \xi \rangle_{L^2(\mathbb{T})} = -\langle 0 \vee (\psi + bg), \xi \rangle_{L^2(\mathbb{T})} = -\langle \psi + bg, \xi \rangle_{L^2(J)}.$$

From here, using the same identity for $\psi + bg_n$, we obtain

$$\begin{aligned}\langle \psi + bg - h, \xi \rangle_{L^2(J)} &= -\langle \psi + bg, \xi \rangle_{L^2(I)} - \langle h, \xi \rangle_{L^2(J)} \\ &= -\lim_{n\to\infty} \langle \psi + bg_n, \xi \rangle_{L^2(I)} - \langle h, \xi \rangle_{L^2(J)} \\ &= \lim_{n\to\infty} \langle \psi + bg_n, \xi \rangle_{L^2(J)} - \langle h, \xi \rangle_{L^2(J)}.\end{aligned}$$

Since $g_n \in \mathcal{B}^{\psi,b}_{M,h}$ for all n, the Cauchy–Schwarz inequality gives

$$|\langle \psi + bg - h, \xi \rangle_{L^2(J)}| = \lim_{n\to\infty} |\langle \psi + bg_n - h, \xi \rangle_{L^2(J)}| \le M \|\xi\|_{L^2(J)}$$

for any $\xi \in \bar{H}^2_0|_J$. The final result is now furnished by employing the density of $\bar{H}^2_0|_J$ in $L^2(J)$ (Proposition 2.4 and Remark 2.1) and the dual characterization of the $L^2(J)$-norm:

$$\|\psi + bg - h\|_{L^2(J)} = \sup_{\substack{\xi \in L^2(J) \\ \|\xi\|_{L^2(J)} \le 1}} |\langle \psi + bg - h, \xi \rangle_{L^2(J)}| = \sup_{\substack{\xi \in \bar{H}^2_0 \\ \|\xi\|_{L^2(J)} \le 1}} |\langle \psi + bg - h, \xi \rangle_{L^2(J)}| \le M. \qquad \square$$

A key property of the solution is that the constraint in (3.2) is necessarily saturated unless $f \in \mathcal{A}^{\psi,b}|_I$.

Lemma 3.1. *If $f \notin \mathcal{A}^{\psi,b}|_I$ and $g \in \mathcal{B}^{\psi,b}_{M,h}$ solves (3.5), then $\|\psi + bg - h\|_{L^2(J)} = M$.*

Proof. To show this, suppose the opposite, i.e. there is $g_0 \in H^2$ solving (3.5) for which we have

$$\|\psi + bg_0 - h\|_{L^2(J)} < M.$$

The last condition means that g_0 is within the interior of $\mathcal{B}^{\psi,b}_{M,h}$, and hence we can define $g^\star := g_0 + \epsilon \delta_g \in \mathcal{B}^{\psi,b}_{M,h}$ for sufficiently small $\epsilon > 0$ and $\delta_g \in H^2$, $\|\delta_g\|_{H^2} = 1$ such that $\operatorname{Re}\langle b\delta_g, \psi + bg_0 - f \rangle_{L^2(I)} < 0$, where the equality case is eliminated by (3.4). By the smallness of ϵ, the quadratic term is negligible, and thus we have

$$\begin{aligned}\|\psi + bg^\star - f\|^2_{L^2(I)} &= \|\psi + bg_0 - f\|^2_{L^2(I)} + 2\epsilon \operatorname{Re}\langle b\delta_g, \psi + bg_0 - f \rangle_{L^2(I)} + \epsilon^2 \|\delta_g\|^2_{L^2(I)} \\ &< \|\psi + bg_0 - f\|^2_{L^2(I)},\end{aligned}$$

which contradicts the minimality of g_0. $\square$

As an immediate consequence of saturation of the constraint, we obtain:

Corollary 3.1. *The requirement $f \in L^2(I) \backslash \mathcal{A}^{\psi,b}|_I$ implies that the formulation of the problem should be restricted to the case $M > 0$.*

Proof. If $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}|_I$ and $M = 0$, the lemma entails that $h \in \mathcal{A}^{\psi,b}|_J$. Then, $h = \psi + bg$ for some $g \in H^2$ and its extension to the whole $\mathbb{D}$ (given, for instance, by Proposition 2.1) uniquely determines $\tilde{g} = h$ without resorting to solution of the bounded extremal problem (3.5), hence independently of f. $\square$

Having established that equality holds in (3.2), we approach (3.5) as a constrained optimization problem following a standard idea of Lagrange multipliers (e.g. [29]) and claim that for a solution g to (3.5) and for some $\tau \in \mathbb{R}$, we must necessarily have

$$\langle \delta_{\tilde{g}}, (\tilde{g} - f) \vee \tau(\tilde{g} - h) \rangle_{L^2(\mathbb{T})} = 0 \tag{3.6}$$

for any $\delta_{\tilde{g}} \in bH^2$ (recall that $\tilde{g} = \psi + bg$ and $\delta_{\tilde{g}} = b\delta_g$ for $\delta_g \in H^2$) which is a condition of tangency of level lines of the objective and constraint functionals. Condition (3.6) can be shown by the same variational argument as in the proof of Lemma 3.1: it must hold true, otherwise we would be able to improve the minimum while still remaining in the admissible set. This motivates us to search for $g \in H^2$ such that, for $\tau \in \mathbb{R}$,

$$[(\psi + bg - f) \vee \tau(\psi + bg - h)] \in (bH^2)^\perp$$

which is equivalent to

$$P_+[\bar{b}(\psi + bg - f) \vee \tau\bar{b}(\psi + bg - h)] = 0. \tag{3.7}$$

Theorem 3.2. *If $f \notin \mathcal{A}^{\psi,b}|_I$ and $M > 0$, the solution to the bounded extremal problem (3.5) is given by*

$$g_0 = (1 + \mu\phi)^{-1} P_+(\bar{b}(f - \psi) \vee (1 + \mu)\bar{b}(h - \psi)), \tag{3.8}$$

where the parameter $\mu > -1$ is uniquely chosen such that $\|\psi + bg_0 - h\|_{L^2(J)} = M$.

Proof. We present here the proof for the case $h \in H^2|_J$, the case $h \in L^2(J)$ is essentially based on the density result of Proposition 2.4 and weak compactness of the unit ball in H^2, it is more technical and given in [8].

First, for simplicity, we assume that $h = 0$. Then, equation (3.7) can be elaborated as follows:

$$\begin{aligned} P_+(\bar{b}(\psi + bg)) + (\tau - 1)P_+(0 \vee \bar{b}(\psi + bg)) &= P_+(\bar{b}f \vee 0), \\ g + P_+(\bar{b}\psi) + (\tau - 1)P_+(0 \vee \bar{b}\psi) + (\tau - 1)\phi g &= P_+(\bar{b}f \vee 0), \\ (1 + \mu\phi)g &= -P_+(\bar{b}(\psi - f) \vee (1 + \mu)\bar{b}\psi), \end{aligned} \tag{3.9}$$

where we introduced the parameter $\mu := \tau - 1 \in \mathbb{R}$.

The Toeplitz operator ϕ, defined as (2.3), is self-adjoint and, according to Proposition 2.2, its spectrum is

$$\sigma(\phi) = [\operatorname{ess\,inf} \chi_J, \operatorname{ess\,sup} \chi_J] = [0, 1], \tag{3.10}$$

hence $\|\phi\| \leq 1$ and the operator $(1 + \mu\phi)$ is invertible on H^2 for $\mu > -1$ allowing to claim that

$$g = -(1 + \mu\phi)^{-1} P_+(\bar{b}(\psi - f) \vee (1 + \mu)\bar{b}\psi). \tag{3.11}$$

This generalizes the result of [3] to the case when solution needs to meet pointwise interpolation conditions. Now, let $h \neq 0$, but assume it to be the restriction to J of some H^2-function.

We write $f = \varrho + \kappa|_I$ for $\kappa \in H^2$ such that $\kappa|_J = h$. Then, the solution to (3.5) is

$$g_0 = \arg\min_{g \in \mathcal{B}^{\psi,b}_{M,h}} \|\psi + bg - f\|_{L^2(I)} = \arg\min_{g \in \tilde{\mathcal{B}}_{M,0}} \|\tilde{\psi} + bg - \varrho\|_{L^2(I)},$$

where $\tilde{\psi} := \psi - \kappa$ and

$$\tilde{\mathcal{B}}_{M,0} := \{g \in H^2 : \|\tilde{\psi} + bg\|_{L^2(J)} \leq M\}.$$

It is easy to see that, due to $\kappa|_J = h$, we have $\tilde{\mathcal{B}}_{M,0} = \mathcal{B}^{\psi,b}_{M,h}$. Therefore, the already obtained results (3.9), (3.11) apply to yield

$$\begin{aligned} (1 + \mu\phi)g_0 &= -P_+(\bar{b}(\tilde{\psi} - \varrho) \vee (1 + \mu)\bar{b}\tilde{\psi}) \\ &= -P_+(\bar{b}(\psi - \kappa - \varrho) \vee (1 + \mu)\bar{b}(\psi - \kappa)) \\ &= P_+(\bar{b}(f - \psi) \vee (1 + \mu)\bar{b}(h - \psi)), \end{aligned} \tag{3.12}$$

from where (3.8) follows. □

Remark 3.1. As it is mentioned in the formulation of Theorem 3.2, for g_0 to be a solution to (3.5), the Lagrange parameter μ has yet to be chosen such that g_0 given by (3.8) satisfies the constraint

$$\|\psi + bg_0 - h\|_{L^2(J)} = M,$$

which makes the well-posedness effective, see Proposition 2.5 and the discussion in the beginning of Section 4. We note that the formal substitution $\mu = -1$ in (3.12) leaves out the constraint on J and leads to the situation $g|_I = \bar{b}(f - \psi)$ that was ruled out initially by the requirement (3.4).

In the situation when $f \in \mathcal{A}^{\psi,b}|_I$, we face an extrapolation problem of holomorphic extension from I inside the disk preserving interior pointwise data. In such a case, $\bar{b}(f - \psi) \in H^2|_I$ and Proposition 2.1 (or any another recovery scheme from [26]) applies to construct the extension g such that $g|_I = \bar{b}(f - \psi)$ and so $\psi + bg$ can be regarded as the solution to the approximation problem if $\|\psi + bg - h\|_{L^2(J)} \leq M$. Otherwise, despite $f \in \mathcal{A}^{\psi,b}|_I$, the cost functional of problem (3.5) cannot be minimized to zero, in this case the solution is $\psi + bg_0$ with g_0

given by (3.8). We note that the derivation of (3.8) was based solely on Lemma 3.1 whose proof still holds true (by hypothesis, $g_0|_I \neq \bar{b}(f-\psi)$ for $g_0 \in \mathcal{B}^{\psi,b}_{M,h}$). This situation, from a geometrical point of view, is nothing but finding a projection of $f \in \mathcal{A}^{\psi,b}|_I$ onto the convex subset $\mathcal{C}^{\psi,b}_{M,h} \subseteq \mathcal{A}^{\psi,b}|_I$.

However, returning back to the realistic case where $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}|_I$, the solution to (3.5) can still be written in an integral form in spirit of the Carleman's formula (2.2) as given by the following result (see also [6] where it was stated for the case $\psi \equiv 0$, $b \equiv 1$).

Proposition 3.1. *For $\mu \in (-1, 0)$, the solution* (3.8) *can be represented as*

$$g_0(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \left(\frac{\Phi(\xi)}{\Phi(z)} \right)^{\alpha} (\bar{b}(f-\psi) \vee \bar{b}(h-\psi))(\xi) \frac{d\xi}{\xi - z}, \quad z \in \mathbb{D}, \tag{3.13}$$

where

$$\Phi(z) = \exp\left\{ \frac{\log \rho}{2\pi i} \int_I \frac{\xi + z}{\xi - z} \, d\xi \right\}, \quad \alpha = -\frac{\log(1+\mu)}{2 \log \rho}, \quad \rho > 1. \tag{3.14}$$

Proof. First of all, we note that the argument of the exponential in (3.14) is an analytic function whose real part has constant boundary value $\log \rho$ supported on I as a convolution with the Schwarz kernel. The function (3.14) satisfies $|\Phi| = \rho \vee 1$ on $\mathbb{T}$ and $|\Phi| > 1$ on $\mathbb{D}$ and so, by the minimum modulus principle for analytic functions, $|\Phi| > 1$ on $\mathbb{D}$ due to the requirement $\rho > 1$.

To show the equivalence of two forms of the solution, one can start from (3.13) and arrive at (3.8) for a suitable choice of the parameters. Indeed, since $\Phi \in H^\infty$, (3.13) implies

$$\Phi^\alpha g_0 = P_+[\Phi^\alpha(\bar{b}(f-\psi) \vee \bar{b}(h-\psi))] \implies P_+(|\Phi|^{2\alpha} g_0) = P_+(\bar{\Phi}^\alpha P_+[\Phi^\alpha(\bar{b}(f-\psi) \vee \bar{b}(h-\psi))]).$$

We can represent

$$P_+[\Phi^\alpha(\bar{b}(f-\psi) \vee \bar{b}(h-\psi))] = \Phi^\alpha(\bar{b}(f-\psi) \vee \bar{b}(h-\psi)) - P_-[\Phi^\alpha(\bar{b}(f-\psi) \vee \bar{b}(h-\psi))]$$

with P_- being the anti-analytic projection defined in Section 2. Since

$$\langle \bar{\Phi}^\alpha P_-[\Phi^\alpha(\bar{b}(f-\psi) \vee \bar{b}(h-\psi))], u \rangle_{L^2(\mathbb{T})} = \langle P_-[\Phi^\alpha(\bar{b}(f-\psi) \vee \bar{b}(h-\psi))], \Phi^\alpha u \rangle_{L^2(\mathbb{T})} = 0$$

for any $u \in H^2$, it follows that

$$P_+(\bar{\Phi}^\alpha P_-[\Phi^\alpha(\bar{b}(f-\psi) \vee \bar{b}(h-\psi))]) = 0$$

and so we deduce

$$P_+(|\Phi|^{2\alpha} g_0) = P_+[|\Phi|^{2\alpha}(\bar{b}(f-\psi) \vee \bar{b}(h-\psi))].$$

Given $\rho > 1$, choose $\alpha > 0$ such that $\rho^{2\alpha} = \frac{1}{1+\mu}$ (this restricts the range $\mu > -1$ to $\mu \in (-1, 0)$). Then, we have $|\Phi|^{2\alpha}|_I = \frac{1}{1+\mu}$, $|\Phi|^{2\alpha}|_J = 1$, and hence

$$P_+\left(\frac{1}{1+\mu} g_0 \vee g_0 \right) = P_+\left(\frac{\bar{b}}{1+\mu}(f-\psi) \vee \bar{b}(h-\psi) \right)$$
$$\implies P_+(g_0 \vee g_0) + \mu P_+(0 \vee g_0) = P_+(\bar{b}(f-\psi) \vee (1+\mu)\bar{b}(h-\psi)),$$

which directly furnishes (3.8). □

3.3 Choice of the interpolation function

Before we proceed with approximation estimates, it is worth discussing the choice of interpolant ψ which up to this point was any H^2-function satisfying the interpolation conditions (3.1). It may look tempting to take advantage of this arbitrariness of the interpolant to improve the solution. However, even though ψ affects the minimizer of the bounded extremal problem (3.5), the choice of the interpolant does not alter the solution to the approximation problem (3.3), i.e. the combination $\tilde{g}_0 = \psi + b g_0$. This result is not surprising at all from physical point of view since ψ is an auxiliary tool which should not affect the solution whose dependence must eventually boil down to given data (measurement related quantities) only: $\{z_k\}_{k=1}^N$, $\{\omega_k\}_{k=1}^N$, f and h. More precisely, we have the following lemma.

Lemma 3.2. *Given arbitrary* $\psi_1, \psi_2 \in H^2$ *satisfying* (3.1), *we have* $\psi_1 + bg_0(\psi_1) = \psi_2 + bg_0(\psi_2)$.

Proof. First of all, we note that the dependence $g_0(\psi)$ is not only due to explicit appearance of ψ in (3.8), but also because the Lagrange parameter μ, in general, has to be readjusted according to ψ, that is, $\mu = \mu(\psi)$ so that

$$\|\psi_k + bg_0(\psi_k) - h\|^2_{L^2(J)} = M^2, \quad k = 1, 2, \tag{3.15}$$

where we mean $g_0(\psi) = g_0(\psi, \mu(\psi))$. Let us denote $\delta_\psi := \psi_2 - \psi_1$, $\delta_\mu := \mu(\psi_2) - \mu(\psi_1)$, $\delta_g := g_0(\psi_2) - g_0(\psi_1)$. Taking difference of both equations (3.15), we have

$$\begin{aligned} &\langle \delta_\psi + b\delta_g, \psi_1 + bg_0(\psi_1) - h\rangle_{L^2(J)} + \langle \psi_2 + bg_0(\psi_2) - h, \delta_\psi + b\delta_g\rangle_{L^2(J)} = 0 \\ &\implies \operatorname{Re}\langle \bar{b}\delta_\psi + \delta_g, \bar{b}\psi_2 + g_0(\psi_2) - \bar{b}h\rangle_{L^2(J)} = \|\delta_\psi + b\delta_g\|^2_{L^2(J)}. \end{aligned} \tag{3.16}$$

On the other hand, the optimality condition (3.6) implies that, for any $\xi \in H^2$,

$$\langle \bar{b}\psi_k + g_0(\psi_k) - \bar{b}f, \xi\rangle_{L^2(I)} = -(1 + \mu(\psi_k))\langle \bar{b}\psi_k + g_0(\psi_k) - \bar{b}h, \xi\rangle_{L^2(J)}, \quad k = 1, 2,$$

and therefore

$$\langle \bar{b}\delta_\psi + \delta_g, \xi\rangle_{L^2(I)} = -(1 + \mu(\psi_1))\langle \bar{b}\delta_\psi + \delta_g, \xi\rangle_{L^2(J)} - \delta_\mu\langle \bar{b}\psi_2 + g_0(\psi_2) - \bar{b}h, \xi\rangle_{L^2(J)}. \tag{3.17}$$

Since $\delta_\psi \in H^2$, due to (3.1), it is zero at each z_j, $j = 1, \dots, N$, and hence factorizes as $\delta_\psi = b\eta$ for some $\eta \in H^2$. This allows us to take $\xi = \bar{b}\delta_\psi + \delta_g \in H^2$ in (3.17) to yield

$$\|\eta + \delta_g\|^2_{L^2(I)} = -(1 + \mu(\psi_1))\|\eta + \delta_g\|^2_{L^2(J)} - \delta_\mu\langle \bar{b}\psi_2 + g_0(\psi_2) - \bar{b}h, \eta + \delta_g\rangle_{L^2(J)}.$$

Note that the inner product term on the right-hand side is real-valued since the others are, and so employing (3.16), we arrive at

$$\|\eta + \delta_g\|^2_{L^2(I)} + (1 + \mu(\psi_1))\|\eta + \delta_g\|^2_{L^2(J)} = -\frac{1}{2}\delta_\mu\|\eta + \delta_g\|^2_{L^2(J)}.$$

Suppose $\delta_\mu \geq 0$, now since $\mu > -1$, the positivity of the left-hand side entails that $\delta_\mu = 0$. Now, if $\delta_\mu \leq 0$, interchanging ψ_1 and ψ_2, we would get

$$\|\eta - \delta_g\|^2_{L^2(I)} + (1 + \mu(\psi_2))\|\eta - \delta_g\|^2_{L^2(J)} = \frac{1}{2}\delta_\mu\|\eta - \delta_g\|^2_{L^2(J)},$$

and so again, since the right-hand side is now negative, it must be that $\delta_\mu = 0$ leading to

$$\|\delta_\psi + b\delta_g\|^2_{L^2(\mathbb{T})} = \|\eta + \delta_g\|^2_{L^2(I)} + \|\eta + \delta_g\|^2_{L^2(J)} = 0,$$

which finishes the proof. □

Even though we cannot quantitatively improve the approximation, some particular choice of the interpolant still might be more beneficial for analytical purposes of the solution formula.

Let us consider

$$\psi(z) = \sum_{k=1}^{N} \psi_k \mathcal{K}(z_k, z) \quad \text{with} \quad \mathcal{K}(z_k, z) := \frac{1}{1 - \bar{z}_k z}, \quad z \in \mathbb{D}. \tag{3.18}$$

We refer to [8, 28] where more details on this interpolant can be found. In particular, the constants can always be uniquely determined from the requirement (3.1), and it is remarkable that $\langle \mathcal{K}(z_k, z), bu\rangle_{L^2(\mathbb{T})} = 0$, $k = 1, \dots, N$, for any $u \in H^2$. The latter is due to the fact that the function $\mathcal{K}(\cdot, \cdot)$ is the reproducing kernel for H^2, meaning that, for any $u \in H^2$, $z_0 \in \mathbb{D}$, point evaluation is given by the inner product

$$u(z_0) = \langle u, \mathcal{K}(z_0, \cdot)\rangle_{L^2(\mathbb{T})}.$$

With this choice of the interpolant, we have $P_+(\bar{b}\psi) = 0$, and thus the solution (3.8) takes the form

$$g_0 = (1 + \mu\phi)^{-1}[P_+(\bar{b}(f \vee h)) + \mu P_+(0 \vee \bar{b}(h - \psi))]. \tag{3.19}$$

We therefore conclude with

Corollary 3.2. *Independently of the choice of* $\psi \in H^2$ *fulfilling* (3.1), *the solution to the approximation problem* (3.3) *is given by*

$$\tilde{g}_0 = \psi + b(1 + \mu\phi)^{-1}[P_+(\bar{b}(f \vee h)) + \mu P_+(0 \vee \bar{b}(h - \psi))]. \tag{3.20}$$

4 Approximation estimates

We would like to stress again that formulas (3.8), (3.13) and (3.19) furnish a solution only in an implicit form with the Lagrange parameter μ still to be chosen such that the solution satisfies the equality constraint in (3.2). As it was mentioned in Remark 3.1, the constraint in $\mathcal{B}^{\psi,b}_{M,h}$ does not enter the solution characterization (3.12) when $\mu = -1$, so as $\mu \searrow -1$ we expect perfect approximation of the given $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}|_I$ at the expense of uncontrolled growth of the quantity

$$M_0(\mu) := \|\psi + bg_0(\mu) - h\|_{L^2(J)} \tag{4.1}$$

according to Propositions 2.4 and 2.5. This is not surprising since the inclusion $\mathcal{B}^{\psi,b}_{M_1,h} \subset \mathcal{B}^{\psi,b}_{M_2,h}$ whenever $M_1 < M_2$ implies that the minimum of the cost functional in equation (3.5) sought over $\mathcal{B}^{\psi,b}_{M_1,h}$ is bigger than that for $\mathcal{B}^{\psi,b}_{M_2,h}$. For devising a feasible for applications solution, a suitable trade-off between values of μ governing the quality of approximation on I and admissible bounds M has to be found. We define the approximation error as

$$e(\mu) := \|\psi + bg_0(\mu) - f\|^2_{L^2(I)}, \tag{4.2}$$

and proceed with establishing a connection between e, M_0 and the Lagrange parameter μ.

The essential result is contained in the following lemma.

Lemma 4.1. *For $\mu > -1$, the following monotonicity properties hold:*

$$\frac{de}{d\mu} > 0, \quad \frac{dM_0^2}{d\mu} < 0. \tag{4.3}$$

Moreover, we have

$$\frac{de}{d\mu} = -(\mu + 1)\frac{dM_0^2}{d\mu}. \tag{4.4}$$

Proof. From (3.8), because ϕ and $(1 + \mu\phi)^{-1}$ commute, derivation yields

$$\begin{aligned} \frac{dg_0}{d\mu} &= -(1 + \mu\phi)^{-2}\phi P_+(\bar{b}(f - \psi) \vee (1 + \mu)\bar{b}(h - \psi)) + (1 + \mu\phi)^{-1}P_+(0 \vee \bar{b}(h - \psi)) \\ \Longrightarrow \quad \frac{dg_0}{d\mu} &= -(1 + \mu\phi)^{-1}[\phi g_0 + P_+(0 \vee \bar{b}(\psi - h))], \end{aligned} \tag{4.5}$$

and thus

$$\begin{aligned} \frac{dM_0^2}{d\mu} &= 2\,\mathrm{Re}\left\langle b\frac{dg_0}{d\mu}, \psi + bg_0 - h\right\rangle_{L^2(J)} \\ &= -2\,\mathrm{Re}\langle(1 + \mu\phi)^{-1}[\phi g_0 + P_+(0 \vee \bar{b}(\psi - h))], \phi g_0 + P_+(0 \vee \bar{b}(\psi - h))\rangle_{L^2(\mathbb{T})} < 0. \end{aligned} \tag{4.6}$$

The inequality here is due to the spectral result (3.10) implying

$$\mathrm{Re}\langle(1 + \mu\phi)^{-1}\xi, \xi\rangle_{L^2(\mathbb{T})} = \langle(1 + \mu\phi)^{-1}\xi, \xi\rangle_{L^2(\mathbb{T})} \geq 0$$

for any $\xi \in H^2$ and $\mu > -1$ whereas the equality in (4.6) would be possible, according to Proposition 2.3, only when $g_0|_J = \bar{b}(h - \psi)$, that is, $M_0 = 0$, the case that was eliminated by Corollary 3.1.

Now, for any $\beta \in \mathbb{R}$, making use of (4.5) again, we compute

$$\begin{aligned} \frac{de}{d\mu} &= 2\,\mathrm{Re}\left\langle \frac{dg_0}{d\mu}, \bar{b}(\psi - f) + g_0\right\rangle_{L^2(I)} \\ &= -2\,\mathrm{Re}\langle(1 + \mu\phi)^{-1}[\phi g_0 + P_+(0 \vee \bar{b}(\psi - h))], (\bar{b}(\psi - f) + g_0) \vee 0\rangle_{L^2(\mathbb{T})} \\ &= -\beta\frac{dM_0^2}{d\mu} - 2\,\mathrm{Re}\,B, \end{aligned}$$

with B given by

$$\begin{aligned} &\langle(1 + \mu\phi)^{-1}[\phi g_0 + P_+(0 \vee \bar{b}(\psi - h))], \beta\phi g_0 + \beta P_+[0 \vee \bar{b}(\psi - h)] + (\bar{b}(\psi - f) + g_0) \vee 0\rangle_{L^2(\mathbb{T})} \\ &= \langle(1 + \mu\phi)^{-1}[\phi g_0 + P_+(0 \vee \bar{b}(\psi - h))], (\bar{b}(\psi - f) + g_0) \vee \beta[\bar{b}(\psi - h) + g_0]\rangle_{L^2(\mathbb{T})} \\ &= \langle b(1 + \mu\phi)^{-1}[\phi g_0 + P_+(0 \vee \bar{b}(\psi - h))], (\psi + bg_0 - f) \vee \beta(\psi + bg_0 - h)\rangle_{L^2(\mathbb{T})}, \end{aligned}$$

where we suppressed the P_+ operator on the right part of the inner product in the second line due to the fact that its left part belongs to H^2.

The choice $\beta = \mu + 1 = \tau$ entails $\operatorname{Re} B = 0$ due to (3.6), and we thus obtain (4.4). Since $\mu + 1 > 0$, equation (4.4) combines with (4.6) to furnish the second inequality in (4.3). □

In particular, equation (4.4) encodes how the decay of the approximation error on I is accompanied by the approximant $\tilde{g}_0 = \psi + bg_0$ departing further away from the given h on J as $\mu \searrow -1$. Increasing behavior of $M_0(\mu)$ near $\mu = -1$ is subject to a rough square-integrability result analogous to [3, Proposition 5].

Proposition 4.1. *The deviation M_0 of the approximant $\tilde{g}_0$ from h on J has moderate growth as $\mu \searrow -1$ so that, for any $-1 < \mu_0 < \infty$,*

$$\int_{-1}^{\mu_0} M_0^2(\mu)\, d\mu < \infty. \tag{4.7}$$

Proof. Integration of (4.4) by parts from μ to μ_0 yields

$$e(\mu_0) - e(\mu) = (\mu + 1)M_0^2(\mu) - (\mu_0 + 1)M_0^2(\mu_0) + \int_{\mu}^{\mu_0} M_0^2(\tau)\, d\tau. \tag{4.8}$$

As it was already mentioned in the beginning of the section, Proposition 2.4 implies that the cost functional goes to 0 when μ decays to -1:

$$e(\mu) \searrow 0 \quad \text{as } \mu \searrow -1. \tag{4.9}$$

We are now going to estimate the behavior of the product $(\mu + 1)M_0^2(\mu)$. First of all, since the constraint is saturated (Lemma 3.1), condition (3.7) implies that

$$\begin{aligned}\langle f - \psi - bg_0, bg_0\rangle_{L^2(I)} &= (1 + \mu)\langle h - \psi - bg_0, -bg_0\rangle_{L^2(J)} \\ &= (1 + \mu)M_0^2 - (1 + \mu)\langle h - \psi - bg_0, h - \psi\rangle_{L^2(J)},\end{aligned}$$

and therefore

$$e^{1/2}(\mu)\|g_0\|_{L^2(I)} \geq |\langle f - \psi - bg_0, bg_0\rangle_{L^2(I)}| \geq (1 + \mu)M_0(M_0 - \|h - \psi\|_{L^2(J)}).$$

Now, since $M_0 \nearrow \infty$ as $\mu \searrow -1$ (because of (4.9) and Proposition 2.4), the first term is dominant, and thus the right-hand side remains positive. Then, because of (4.9) and finiteness of $\|g_0\|_{L^2(I)}$ (by the triangle inequality, $\|g_0\|_{L^2(I)} \leq e^{1/2}(\mu) + \|\psi - f\|_{L^2(I)}$), we conclude that

$$(\mu + 1)M_0^2 \searrow 0 \quad \text{as } \mu \searrow -1,$$

which allows us to deduce (4.7) from (4.8). □

It turns out that we can obtain explicit expressions for $e(\mu)$ and $M_0(\mu)$. The latter would yield the unique value of the parameter μ by the inverse function theorem applicable due to the monotonicity result (4.6).

It is convenient to introduce the following auxiliary quantity:

$$\xi(\mu) := \phi g_0(\mu) + P_+(0 \vee \bar{b}(\psi - h))$$

that enters equation (4.5). The main results will be obtained in terms of

$$\xi_0 := \xi(0) = \phi(P_+(\bar{b}(f - \psi) \vee \bar{b}(h - \psi))) - P_+(0 \vee \bar{b}(h - \psi)). \tag{4.10}$$

Theorem 4.1. *For $|\mu| < 1$, the quantities (4.1)–(4.2) can be computed as*

$$M_0^2(\mu) = M_0^2(0) - \sum_{k=0}^{\infty} (-1)^k (k + 2)F(k)\mu^{k+1} \tag{4.11}$$

and

$$e(\mu) = e(0) + 2\sum_{k=0}^{\infty} (-1)^k F(k)\mu^{k+1} + \sum_{k=1}^{\infty} (-1)^k k[F(k) - F(k - 1)]\mu^{k+1}, \tag{4.12}$$

where $F(k) := \langle \phi^k \xi_0, \xi_0\rangle_{L^2(\mathbb{T})}$, $k \in \mathbb{N}_+$.

Proof. Consider, for $k \in \mathbb{N}_+$, $\mu > -1$,

$$A_k(\mu) := \langle (1+\mu\phi)^{-k}\phi^{k-1}\xi(\mu), \xi(\mu)\rangle_{L^2(\mathbb{T})}.$$

Since $\xi'(\mu) = \phi \frac{dg_0}{d\mu} = -(1+\mu\phi)^{-1}\phi\xi(\mu)$ (according to (4.5)), it follows that

$$A_k'(\mu) = -k\langle (1+\mu\phi)^{-k-1}\phi^k\xi(\mu), \xi(\mu)\rangle_{L^2(\mathbb{T})} - \langle (1+\mu\phi)^{-k-1}\phi^k\xi(\mu), \xi(\mu)\rangle_{L^2(\mathbb{T})}$$
$$- \langle (1+\mu\phi)^{-k}\phi^{k-1}\xi(\mu), (1+\mu\phi)^{-1}\phi\xi(\mu)\rangle_{L^2(\mathbb{T})},$$

and we thus arrive at the infinite-dimensional linear dynamical system

$$\begin{cases} A_k'(\mu) = -(k+2)A_{k+1}(\mu), \\ A_k(0) = \langle \phi^{k-1}\xi_0, \xi_0\rangle_{L^2} =: F(k-1), \end{cases} \quad k \in \mathbb{N}_+. \tag{4.13}$$

Introduce the matrix $\mathcal{M}$ whose powers are upper-diagonal with evident structure

$$\mathcal{M} = \begin{pmatrix} 0 & -3 & 0 & 0 & \dots \\ 0 & 0 & -4 & 0 & \dots \\ 0 & 0 & 0 & -5 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \mathcal{M}^2 = \begin{pmatrix} 0 & 0 & (-3)(-4) & 0 & \dots \\ 0 & 0 & 0 & (-4)(-5) & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \dots,$$

which makes the matrix $e^{\mathcal{M}}$ easily computable. Then, due to such a structure, system (4.13) is readily solvable, but of particular interest is the first component of the solution vector

$$A_1(\mu) = \sum_{k=1}^{\infty} [e^{\mathcal{M}\mu}]_{1,k} F(k-1) = \sum_{k=0}^{\infty} (-1)^k \frac{(k+2)!}{2}\frac{\mu^k}{k!} F(k),$$

where the series converges for $|\mu| < 1$ since $F(k)$ is bounded by $\|\xi_0\|_{H^2}^2 = A_1(0) = F(0)$, as the Toeplitz operator ϕ is a contraction: $F(k)$ slowly decays to zero with k (see also plots and discussion at the end of Section 6).

On the other hand, observe that, due to (4.6),

$$A_1(\mu) = -\frac{1}{2}\frac{dM_0^2}{d\mu}$$

and thus

$$\frac{dM_0^2}{d\mu} = -\sum_{k=0}^{\infty} (-1)^k (k+1)(k+2)\mu^k F(k). \tag{4.14}$$

Finally, termwise integration of (4.14) and use of (4.4) followed by rearrangement of terms furnish the results (4.11)–(4.12). □

Remark 4.1. Note that when $\psi \equiv 0$, $h \equiv 0$, it is seen that (4.14) can be obtained directly from (3.8) and (4.6) which now reads

$$\frac{dM_0^2}{d\mu} = -2\,\mathrm{Re}\langle (1+\mu\phi)^{-3}\phi^2 P_+(\bar{b}f \vee 0), P_+(\bar{b}f \vee 0)\rangle_{L^2(\mathbb{T})}.$$

Indeed, the result follows since a Neumann series defining an analytic function for $|\mu| < 1$ is differentiable:

$$(1+\mu\phi)^{-1} = \sum_{k=0}^{\infty} (-1)^k \mu^k \phi^k \implies (1+\mu\phi)^{-3} = \frac{1}{2}\sum_{k=0}^{\infty} (-1)^k (k+1)(k+2)\mu^k\phi^k.$$

5 Stability analysis

The issue to be discussed here is linear stability of the solution (3.3) with respect to all the physical components that the expression (3.8) involves explicitly and implicitly. In practice, functions f, h are typically obtained by further interpolating discrete boundary data and hence may vary depending on the interpolation method, measurement positions $\{z_j\}_{j=1}^N$ are usually known with a small error and pointwise data $\{\omega_j\}_{j=1}^N$ are necessarily subject to a certain noise. Therefore, we assume that the boundary data f, h are slightly perturbed by $\delta_f \in L^2(I)$, $\delta_h \in L^2(J)$ and internal data $\{\omega_j\}_{j=1}^N$ with measurement positions $\{z_j\}_{j=1}^N$ by complex vectors $\boldsymbol{\delta_\omega}$, $\boldsymbol{\delta_z} \in \mathbb{C}^N$, respectively. Varying one of the quantities while the others are kept fixed, we are going to estimate separately the linear effects of such perturbations on the solution $\tilde{g}_0 = \psi + bg_0$ given by (3.3), denoting the induced deviations as $\delta_{\tilde{g}}$.

Proposition 5.1. *For $\mu > -1$, $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}|_I$, $h \in L^2(J)$, and small enough data perturbations $\delta_f \in L^2(I)$, $\delta_h \in L^2(J)$, $\boldsymbol{\delta_\omega}$, $\boldsymbol{\delta_z} \in \mathbb{C}^N$, the following estimates hold:*

$$\|\delta_{\tilde{g}}\|_{H^2} \le m_1\left(1 + \frac{m_1 M^2}{m_0\|\xi\|_{H^2}^2}\right)\|\delta_f\|_{L^2(I)}, \tag{1}$$

$$\|\delta_{\tilde{g}}\|_{H^2} \le \left[(1 + m_1(1+\mu))\left(1 + \frac{m_1 M^2}{m_0\|\xi\|_{H^2}^2}\right) - 1\right]\|\delta_h\|_{L^2(J)}, \tag{2}$$

$$\|\delta_{\tilde{g}}\|_{H^2} \le (1 + |\mu| m_1)\left(1 + \frac{m_1 M^2}{m_0\|\xi\|_{H^2}^2}\right)\max_{j=1,\dots,N}\left\|\prod_{\substack{k=1\\k\neq j}}^{N}\frac{z - z_k}{z_j - z_k}\right\|_{H^2}\|\boldsymbol{\delta_\omega}\|_{l^1}, \tag{3}$$

$$\|\delta_{\tilde{g}}\|_{H^2} \le \left(1 + \frac{m_1 M^2}{m_0\|\xi\|_{H^2}^2}\right)\left(C_\mu^{(1)}\|\delta_b\|_{H^\infty} + C_\mu^{(2)}\|\delta_\psi\|_{H^2}\right), \tag{4}$$

where

$$\xi := P_+(0 \vee (g_0 + \bar{b}(\psi - h))), \qquad m_0 := \min\{(1+\mu)^{-1}, 1\}, \quad m_1 := \max\{(1+\mu)^{-1}, 1\}, \tag{5.1}$$
$$C_\mu^{(1)} := m_1(\|f \vee h\|_{L^2(\mathbb{T})} + |\mu|\|h - \psi\|_{L^2(J)}), \quad C_\mu^{(2)} := 1 + |\mu| m_1,$$

and

$$\|\delta_b\|_{H^\infty} \le 2\max_{j=1,\dots,N}\|(z - z_j)^{-1}\|_{H^\infty}\|\boldsymbol{\delta_z}\|_{l^1},$$

$$\|\delta_\psi\|_{H^2} \le 2\max_{j=1,\dots,N}|\omega_j|\max_{j=1,\dots,N}\left\|\prod_{\substack{m=1\\m\neq j}}^{N}(z - z_m)\right\|_{H^2}\max_{j=1,\dots,N}\sum_{\substack{k=1\\k\neq j}}^{N}|z_j - z_k|^{-1}\left(\min_{j=1,\dots,N}\prod_{\substack{k=1\\k\neq j}}^{N}|z_j - z_k|\right)^{-1}\|\boldsymbol{\delta_z}\|_{l^1}.$$

Proof. When the quantities entering the solution (3.8) vary, the overall variation of the solution δ_g will consist of parts δ_{g_0} entering the solution formula explicitly as well as those coming from the change of the norm of g_0 on J which leads to readjustment δ_μ of the Lagrange parameter so that the quantity

$$M_0^2(\mu) = \|\psi + bg_0(\mu) - h\|_{L^2(J)}^2$$

remains equal to the same given value M^2. For the sake of brevity, we are going to use the notations ξ, m_0 and m_1 introduced in (5.1) to denote certain quantities entering common estimates. The spectral bounds (3.10) for $\mu > -1$ imply

$$\begin{aligned} &\sigma(1 + \mu\phi) \ge \min\{1 + \mu, 1\}, &&\sigma(1 + \mu\phi) \le \max\{1 + \mu, 1\} \\ \Longrightarrow\quad &\|(1 + \mu\phi)^{-1}\| \le \max\{(1+\mu)^{-1}, 1\}, &&\|(1 + \mu\phi)^{-1}\| \ge \min\{(1+\mu)^{-1}, 1\}, \end{aligned}$$

and so, in particular,

$$\operatorname{Re}\langle(1 + \mu\phi)^{-1}\xi, \xi\rangle_{L^2(\mathbb{T})} \ge m_0\|\xi\|_{H^2}^2.$$

Then, the connection between the change δ_{M^2} of $M_0^2(\mu)$ and δ_μ can be established from the strict monotonicity (4.6) of $M_0(\mu)$ which allows the following estimate by inversion:

$$\delta_\mu = \frac{\delta_{M^2}}{(M_0^2(\mu))'} = -\frac{\delta_{M^2}}{2\operatorname{Re}\langle(1 + \mu\phi)^{-1}\xi, \xi\rangle_{L^2(\mathbb{T})}} \Longrightarrow |\delta_\mu| \le \frac{|\delta_{M^2}|}{2m_0\|\xi\|_{H^2}^2}. \tag{5.2}$$

Note that the bound on the right-hand side is finite due to the fact that $\|\xi\|_{H^2} > 0$ which holds unless $M_0(\mu) = 0$, the situation that was initially ruled out by Corollary 3.1. Discussion on an a priori estimate of $\|\xi\|_{H^2}$ will be given in Remark 5.1.

Following this strategy, we embark on consecutive proof of the results (1)–(4).

Result (1). This is the simplest case, the variation of $M_0^2(\mu)$ is induced only by change of g_0. Namely,

$$\delta_{M^2} = 2\,\mathrm{Re}\langle \psi + bg_0(\mu) - h, b\delta_{g_0}(\mu)\rangle_{L^2(J)}, \tag{5.3}$$

where

$$\delta_{g_0} = (1+\mu\phi)^{-1}P_+(\bar{b}\delta_f \vee 0). \tag{5.4}$$

Application of the Cauchy–Schwarz inequality to (5.3) yields

$$|\delta_{M^2}| \le 2M_0(\mu)\|(1+\mu\phi)^{-1}\|\|P_+(\bar{b}\delta_f \vee 0)\|_{L^2(\mathbb{T})} \le 2M_0(\mu)m_1\|\delta_f\|_{L^2(I)}$$

and hence, by (5.2),

$$|\delta_\mu| \le \frac{m_1 M_0(\mu)}{m_0\|\xi\|^2_{H^2}}\|\delta_f\|_{L^2(I)}.$$

Now since $\delta_{\tilde{g}} = b\delta_g$, due to (4.5), we have

$$\delta_{\tilde{g}} = b\delta_{g_0} - b(1+\mu\phi)^{-1}P_+(0 \vee (g_0 + \bar{b}(\psi - h)))\delta_\mu, \tag{5.5}$$

from where using (5.4) we deduce inequality (1).

Result (2). This is totally analogous to the previous result except that now we have

$$\delta_{M^2} = 2\,\mathrm{Re}\langle \psi + bg_0(\mu) - h, b\delta_{g_0}(\mu) - \delta_h\rangle_{L^2(J)}$$

with

$$\delta_{g_0} = (1+\mu\phi)^{-1}P_+(0 \vee (1+\mu)\bar{b}\delta_h).$$

Therefore,

$$|\delta_{M^2}| \le 2M_0(\mu)[1+(1+\mu)m_1]\|\delta_h\|_{L^2(J)} \implies |\delta_\mu| \le \frac{M_0(\mu)[1+(1+\mu)m_1]}{m_0\|\xi\|^2_{H^2}}\|\delta_h\|_{L^2(J)}.$$

Feeding this in relation (5.5), which still holds in this case, gives

$$\|\delta_{\tilde{g}}\|_{H^2} \le m_1\left(1+\mu+\frac{[1+(1+\mu)m_1]M^2}{m_0\|\xi\|^2_{H^2}}\right)\|\delta_h\|_{L^2(J)},$$

that is exactly a rewording of estimate (2).

Results (3)–(4). The proofs are routine which differ from (2) only technically and can be found in detail in [8]. □

Remark 5.1. The quantity ξ introduced in (5.1) enters the results (1)–(4) of Proposition 5.1 and should be bounded away from zero. This fact, however, follows from Proposition 2.3 and Corollary 3.1. Moreover, the norm of ξ can be a priori estimated as

$$\|\xi\|_{H^2} \ge \frac{1}{|\mu|}(M - \|\psi - h + bP_+(\bar{b}(f \vee h))\|_{L^2(J)}) \tag{5.6}$$

by applying the triangle inequality for $L^2(J)$-norm of the quantity

$$\psi + bg_0 - h = \psi - h + bP_+(\bar{b}(f \vee h)) + \mu bP_+(0 \vee (\bar{b}(h-\psi) - g_0)),$$

which is a consequence of (3.12). Of course, estimate (5.6) is useful only under assumption

$$\|\psi - h + bP_+(\bar{b}(f \vee h))\|_{L^2(J)} < M, \tag{5.7}$$

but we do not include it in the formulation of Proposition 5.1, since this inequality can be achieved without imposing any restriction on given boundary data f and h or increasing the bound M: since, according to Lemma 3.2, the choice of ψ does not affect the solution $\tilde{g}_0$ whose stability is investigated, one can consider another instance of bounded extremal problem, now formulated for $\psi \in H^2$ meeting pointwise constraints (3.1) and approximating $h - bP_+(\bar{b}(f \vee h)) \in L^2(J)$ on J sufficiently closely (with precision M) with a finite bound on I without any additional information (meaning that for such a problem roles I and J are swapped and $h = 0$). To be more precise, given arbitrary $\psi_0 \in H^2$ satisfying pointwise interpolation conditions (3.1) (for instance, one can use (3.18)), we represent $\psi = \psi_0 + b\Psi$ and thus search for an approximant $\Psi \in H^2$ to "f" $= \bar{b}(h - \psi_0) - P_+(\bar{b}(f \vee h)) \in L^2(J)$ such that $\|\Psi\|_{L^2(I)} = \tilde{M}$ for arbitrary $\tilde{M} \in (0, \infty)$. We also note that in the case of reduction to the previously considered problem with no pointwise data imposed ([3, 6]), i.e. when $\psi \equiv 0$ and $b \equiv 1$, one does not have flexibility of varying the interpolant. However, the stability estimates still persist in the region of interest (that is, for $-1 < \mu < 0$) since condition (5.7) is fulfilled as long as $\mu < 0$ due to (3.8) evaluated at $\mu = 0$ and (4.3).

Remark 5.2. Results (3)–(4) technically show stability in terms of finite pointwise data sets $\{\omega_j\}_{j=1}^N$, $\{z_j\}_{j=1}^N$ in l^1-norm. However, by the equivalence of norms in finite dimensions, the same results, but with different bounds, also hold for l^p-norms, for any $p \in \mathbb{N}_+$ and $p = \infty$.

6 Numerical illustrations and algorithmic aspects

In the present section we would like to illustrate numerically the efficiency of our series expansion method.

6.1 Numerical set-up

First of all, without loss of generality, choose $J = \{e^{i\theta} : \theta \in [-\theta_0, \theta_0]\}$ for some fixed $\theta_0 \in (0, 2\pi)$. In order to invert the Toeplitz operator in (3.8) in a computationally efficient way, we project equation (3.12) onto a finite-dimensional (truncated) Fourier basis $\{z^{k-1}\}_{k=1}^Q$ for large enough $Q \in \mathbb{N}_+$ and look for approximate solution in the form

$$g(z) = \sum_{k=1}^{Q} g_k z^{k-1}. \tag{6.1}$$

Introducing, for $m, k \in \{1, \dots, Q\}$,

$$A_{k,m} := \begin{cases} \frac{\sin(m-k)\theta_0}{\pi(m-k)}, & m \neq k, \\ \frac{\theta_0}{\pi}, & m = k, \end{cases} \qquad A := [A_{k,m}]_{k,m=1}^Q, \tag{6.2}$$
$$s_k := \langle (\bar{b}(f - \psi) \vee (1 + \mu)\bar{b}(h - \psi)), e^{i(k-1)\theta} \rangle_{L^2(0,2\pi)}, \quad \boldsymbol{s} := [s_k]_{k=1}^Q,$$

the projection equation

$$\langle (1 + \mu\phi)g - P_+(\bar{b}(f - \psi) \vee (1 + \mu)\bar{b}(h - \psi)), z^{k-1} \rangle_{L^2(\mathbb{T})} = 0$$

becomes the vector equation (if we employ 1 to denote the identity $Q \times Q$ matrix)

$$(1 + \mu A)\boldsymbol{g} = \boldsymbol{s}, \quad \boldsymbol{g} := [g_k]_{k=1}^Q$$

with a real symmetric Toeplitz matrix $(1 + \mu A)$ which is computationally cheap to invert: depending on the algorithm, asymptotic complexity of inversion may be as low as $\mathcal{O}(Q \log^2 Q)$ (for instance, see [12]).

Now, in order to numerically demonstrate the monotonicity results (4.3) for e and M_0 with respect to the parameter μ and to compare the behavior with that of series expansions (4.11)–(4.12), we run simulation for the following set of data. We choose $N = 5$, $\theta_0 = \frac{\pi}{3}$, and

$$f(\theta) = f_0(\theta) + \frac{0.5}{\exp(i\theta) - 0.4 - 0.3i}, \quad f_0(\theta) := \exp(5i\theta) + \exp(2i\theta) + 1 \in \mathcal{A}^{\psi,b}$$

(obviously, $f \in L^2(I)$ does not extend inside the disk as an H^2-function). Further, f_0 is the restriction of the

z	ω
$0.5 + 0.4i$	$0.9852 + 0.3752i$
$-0.3 + 0.3i$	$1.0097 - 0.1897i$
$0.2 + 0.6i$	$0.7811 + 0.2362i$
$0.2 - 0.5i$	$0.8328 - 0.1852i$
$0.8 - 0.1i$	$1.9069 - 0.3584i$

Table 1. Interior pointwise data.

function $z^5 + z^2 + 1$ satisfying pointwise interpolation conditions (3.1) for points $\{z_j\}_{j=1}^5$ and values $\{\omega_j\}_{j=1}^5$ chosen as given in Table 1. We also take $h \in L^2(J)$ as

$$h(\theta) = \frac{1}{\exp(i\theta) - 0.5i}.$$

Based on the points $\{z_j\}_{j=1}^5$, we construct the Blaschke product according to (2.1) with the choice of constant $\phi_0 = 0$ (obviously, final physical results should not depend on a choice of this auxiliary parameter which is also clear from the solution formula (3.20)). The interpolant ψ was chosen as (3.18). The series expansions (4.11)–(4.12) are straightforward to evaluate numerically since $F(k)$ involves the quantity ξ_0 given by (4.10). There, the projections P_+ are computed by performing non-negative-power expansions as (6.1) whereas ϕ^k is simply the iterative multiplication of the first Q Fourier coefficients of ξ_0 by the Toeplitz operator matrix (6.2). Such iterations are extremely fast to compute once the matrix A is diagonalized.

In Figure 1, we investigate the change of deviation of the series expansion from the solution computed numerically (which is taken as a reference in this case, see the discussion in the next subsection) as more terms are taken into account in the expansions.

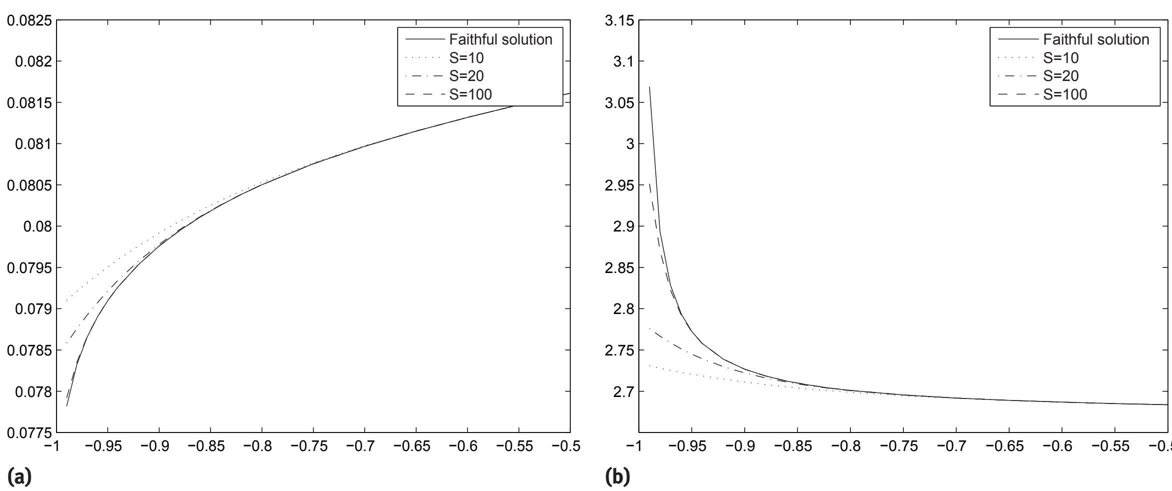

Figure 1. Relative approximation error on I (left) and relative discrepancy error on J (right).

6.2 Suggested computational algorithm

Even though Figure 1 shows good accuracy of approximation $e(\mu)$ and $M_0^2(\mu)$ from the series expansions (4.11)–(4.12), it is clear, by the nature of such expansions, that the convergence slows down as μ gets closer to -1, and hence the number of terms in the series should be increased dramatically. Nevertheless, as it was mentioned, the quantities $F(k)$ are very cheap to compute. It remains only to estimate S, that is, the number of terms in the series for the accurate approximation of $e(\mu)$ and $M_0^2(\mu)$, but it suffices to perform such a calibration only once, namely, for the lowest value of μ in the computational range. This suggests one of possible computational strategies that consists of the following steps:

(1) Decide on the lowest value of the Lagrange parameter μ_0 by checking the approximation rate computed from solving system (6.1). The quantity $e(\mu_0)$ will then be the best approximation rate on I.
(2) Determine the number of terms S by comparing the approximation rate with that evaluated from the expansion (4.12) for $e(\mu_0)$.
(3) Fix S, precompute the values $F(k)$, $k = 1, \dots, S$. Vary the parameter μ and evaluate the approximation and blow-up rates from the expansions (4.11)–(4.12) in order to find a suitable trade-off.

7 Conclusions

Motivated by solving an overdetermined Cauchy problem for the Laplace equation, we have introduced and solved a new bounded extremal problem. It extends the one of best norm-constrained approximation of a given function on a subset of the circle by the trace of an H^2-function [3, 6] to the case where additional pointwise constraints are imposed inside the unit disk. Such an approach makes it possible to take into account outlying measurements rather than discarding them when constructing the interpolation functions u_0, ω_0 or otherwise artificially modifying the boundary of the domain into a less regular one.

While studying this bounded extremal problem, we obtained new results which also apply to the original problem without pointwise constraints, that would be a particular case with $b = 1$, $\psi = 0$ in (3.3). The main new result is a method of direct evaluation of the approximation characteristics in terms of a Lagrange parameter. With an extra argument, the method can be used to obtain the asymptotic estimates for quantities governing the approximation quality that complement those in [5]. In this direction, some technical estimates are available in [8]. More formal and general results were obtained and might be published in a companion paper focussing more on function-theoretical rather than practical aspects of the problem where also infinite number of interior pointwise constraints would be considered.

The new series expansion method was numerically demonstrated to be very efficient especially beyond the asymptotic regime. It thus makes redundant solving multiple instances of the bounded extremal problem iteratively in order to find the Lagrange parameter value corresponding to a suitable trade-off between the approximation rate on the given boundary subset and the solution growth on its complement.

We also performed a number of linear estimates thereby filling a gap of stability analysis for bounded extremal problems. Even without presence of pointwise data, the only available result so far, to our knowledge, was a proof of continuity of the solution with respect to approximated function without additional data ($h = 0$; see [11]).

The suggested method being essentially iteration-free can yield a further iterative computational strategy. One can think of restarting the procedure taking the computed solution as a new reference function h and a reduced value of M.

Acknowledgment: The authors would like to thank Laurent Baratchart for suggesting a valuable possibility of incorporating pointwise data and his contribution to the proof of Theorem 3.1.

References

[1] L. Aizenberg, *Carleman's Formulas in Complex Analysis*, Math. Appl. (Dordrecht) 244, Kluwer, Dordrecht, 1993.
[2] G. Alessandrini, Examples of instability in inverse boundary-value problems, *Inverse Problems* **13** (1997), 887–897.
[3] D. Alpay, L. Baratchart and J. Leblond, Some extremal problems linked with identification from partial frequency data, in: *Analysis and Optimization of Systems: State and Frequency Domain Approaches for Infinite-Dimensional Systems* (Sophia-Antipolis 1992), Lecture Notes in Control and Inform. Sci. 185, Springer, Berlin (1993), 563–573.
[4] L. Baratchart, Y. Fischer and J. Leblond, Dirichlet/Neumann problems and Hardy classes for the planar conductivity equation, *Complex Var. Elliptic Equ.* **59** (2014), no. 4, 504–538.

[5] L. Baratchart, J. Grimm, J. Leblond and J. Partington, Asymptotic estimates for interpolation and constrained approximation in H^2 by diagonalization of Toeplitz operators, *Integral Equations Operator Theory* **45** (2003), 269–299.

[6] L. Baratchart and J. Leblond, Hardy approximation to L^p functions on subsets of the circle with $1 \leq p < \infty$, *Constr. Approx.* **14** (1998), 41–56.

[7] L. Baratchart, J. Leblond and J. Partington, Hardy approximation to L^∞ functions on subsets of the circle, *Constr. Approx.* **12** (1996), 423–436.

[8] L. Baratchart, J. Leblond and D. Ponomarev, Constrained optimization in classes of analytic functions with prescribed pointwise values, preprint (2015), http://arxiv.org/abs/1401.7633.

[9] A. L. Bukhgeim, Extension of solutions of elliptic equations from discrete sets, *J. Inverse Ill-Posed Probl.* **1** (1993), no. 1, 17–32.

[10] S. Chaabane, I. Fellah, M. Jaoua and J. Leblond, Logarithmic stability estimates for a Robin coefficient in 2D Laplace inverse problems, *Inverse Problems* **20** (2004), 49–57.

[11] S. Chaabane, M. Jaoua and J. Leblond, Parameter identification for Laplace equation and approximation in Hardy classes, *J. Inverse Ill-Posed Probl.* **11** (2003), 33–57.

[12] G. Codevico, G. Heinig and M. Van Barel, A superfast solver for real symmetric Toeplitz systems using real trigonometric transformations, *Numer. Linear Algebra Appl.* **12** (2005), no. 8, 699–713.

[13] L. Debnath and P. Mikusinski, *Introduction to Hilbert Spaces with Applications*, Academic Press, Boston, 1990.

[14] L. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Pure Appl. Math. 49, Academic Press, New York, 1972.

[15] P. L. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[16] Y. Fischer, J. Leblond, J. Partington and E. Sincich, Bounded extremal problems in Hardy spaces for the conjugate Beltrami equation in simply connected domains, *Appl. Comput. Harmon. Anal.* **31** (2011), 264–285.

[17] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[18] G. M. Goluzin and V. I. Krylov, Generalized Carleman formula and its application to analytic continuation of functions, *Mat. Sb.* **40** (1933), 144–149.

[19] J. Helsing and B. T. Johansson, Fast reconstruction of harmonic functions from Cauchy data using integral equation techniques, *Inverse Probl. Sci. Eng.* **18** (2010), no. 3, 381–399.

[20] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, 1962.

[21] M. Jaoua, J. Leblond and M. Mahjoub, Robust numerical algorithms based on analytic approximation for the solution of inverse problems in annular domains, *IMA J. Appl. Math.* **74** (2009), 481–506.

[22] M. G. Krein and P. Y. Nudel'man, Approximation of $L_2(\omega_1, \omega_2)$ functions by minimum energy transfer functions of linear systems, *Problemy Peredachi Informatsii* **11** (1975), no. 2, 37–60.

[23] P. Lax, *Functional Analysis*, Wiley, Chichester, 2002.

[24] J. Leblond, J. Partington and E. Pozzi, Best approximation problems in Hardy spaces and by polynomials, with norm constraints, *Integral Equations Operator Theory* **75** (2013), no. 4, 491–516.

[25] N. K. Nikolski, *Operators, Functions and Systems: An Easy Reading. Volume 1: Hardy, Hankel and Toeplitz*, American Mathematical Society, Providence, 2001.

[26] D. J. Patil, Representation of H^p-functions, *Bull. Amer. Math. Soc.* **78** (1972), no. 4, 617–620.

[27] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New-York, 1982.

[28] S. V. Schvedenko, Interpolation in some Hilbert spaces of analytic functions, *Mat. Zametki* **15** (1974), no. 1, 101–112.

[29] P. Varaia, *Notes on Optimization*, Van Nostrand Reinhold, New York, 1972.