A generalised time-evolution model for the sliding punch problem with wear

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Résumé — For a classical contact problem with wear, we consider a newly proposed model where a relation between the pressure and the wear rate is due to a combination of fractional calculus and relaxation effects. The stationary pressure profile and the rate of stabilisation are studied for practically interesting cases of constant and oscillatory loads in time. Numerical simulations reveal differences in a qualitative behavior depending on the model parameters. The proposed model is expected to serve as a simple solvable model for capturing complicated phenomena appearing in complex contact scenarios.
Mots clés — contact mechanics, fractional calculus, structure dynamics.

1 Introduction

Wear of material is known to be a complicated process and its study has produced numerous empirical models in attempt to fit experimental data for particular settings. One classical model of wear is due to Archard [3]. According to this model, the wear rate is proportional to the load with a power-law dependence. This often reduces to a linear relation between the wear rate and the pressure [15, Sect. 17.2–17.3] leading to several effectively solvable tribological models for sliding of an indented punch [5, 6, 7, 8, 11, 18].

We explore a larger class of such models based on a more general relation between the wear rate and the contact pressure by accounting for non-local (temporal) dependencies for a particular setting. Namely, we consider the following instance of the two-dimensional punch problem : a rigid wearable punch, subject to a given normal (vertical) load, slides on a thick elastic layer (modelled as a half-plane), with a prescribed constant speed. The contact area is assumed to be fixed. The described set-up is illustrated in Figure 1.

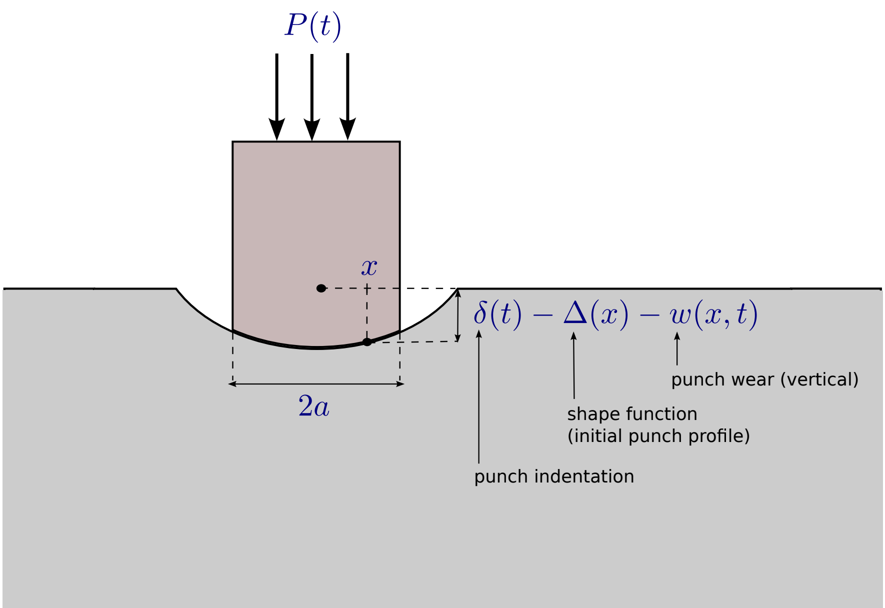

FIGURE 1 – Geometry of the problem. A balance of displacements under the punch is governed by the equation : $\eta p(x,t) + \int_{-a}^{a} K(x-\xi)\, p(\xi,t)\, \mathrm{d}\xi = \delta(t) - \Delta(x) - w(x,t)$, $x \in (-a,a)$, $t \geq 0$.

Since the layer is homogeneous and material wear occurs on the interface, the problem can be effectively described by one-dimensional integral equations. More precisely, according to [1, 5, 6, 7, 9, 12, 16], the pressure under the punch satisfies the following equation for displacements

$$\eta p(x,t)+\int_{-a}^{a} K(x-\xi)\, p(\xi,t)\,\mathrm{d}\xi=\delta(t)-w[p](x,t)-\Delta(x),\quad x\in(-a,a),\quad t\geq 0, \tag{1}$$

and the force equilibrium condition

$$\int_{-a}^{a} p(x,t)\,\mathrm{d}x=P(t),\quad t\geq 0. \tag{2}$$

The contact load $P(t)$ here will be considered constant $P(t)=P_0$ or oscillatory, in particular,

$$P(t)=P_0-P_\Delta+P_\Delta\cos(\omega t) \tag{3}$$

for some given constants P_0, P_Δ, $\omega>0$.

The interval $(-a,a)$ corresponds to the contact area under the punch, $K(x)$ is a kernel function of the "pressure-to-displacement" operator. This operator stems from the Green's function pertinent to a given geometry. Namely, since we model the thick layer as a half-plane, we have

$$K(x):=-\log|x|+C_K, \tag{4}$$

for some constant $C_K>\log a$.

The first term on the left-hand side of (1) accounts for the additional deformation due to the presence of a coating or to model surface roughness [2, 11]. The strength of this effect is measured by the constant $\eta>0$.

The initial punch profile $\Delta(x)$ is a known function whereas the punch indentation $\delta(t)$ is a function of only time-variable t. Its initial value $\delta(0)$ can be found from solving

$$\eta p(x,0)+\int_{-a}^{a} K(x-\xi)\, p(\xi,0)\,\mathrm{d}\xi=\delta(0)-\Delta(x),\quad x\in(-a,a), \tag{5}$$

and requiring that $\int_{-a}^{a} p(x,0)\,\mathrm{d}x=P_0$, as follows from (1) and (2), respectively, evaluated at $t=0$. Finally, $w[p](x,t)$ is the wear term which is an operator acting on the contact pressure $p(x,t)$. Following the discussion in [13, Sec. 3], we take it to be

$$w[p](x,t)=-\nu\mu^{1/\alpha-1}\int_0^t \mathcal{E}_\alpha\left(\mu^{1/\alpha}(t-\tau)\right) p(x,\tau)\,\mathrm{d}\tau,\quad t\geq 0, \tag{6}$$

where $\alpha\in(0,2)$, $\mu>0$ are constants and the special function $\mathcal{E}_\alpha$ is defined as

$$\mathcal{E}_\alpha(t):=\frac{\alpha}{t}\sum_{k=1}^{\infty}\frac{(-1)^k k t^{\alpha k}}{\Gamma(\alpha k+1)},\quad t>0,\quad \alpha>0, \tag{7}$$

with Γ being the Gamma function. To show that this is a meaningful generalisation of the classical wear relation, let us consider some particular cases that it encompasses. Namely, on the one hand, by taking $\alpha=1$, we obtain

$$w[p](x,t)=\nu\int_0^t e^{-\mu(t-\tau)} p(x,\tau)\,\mathrm{d}\tau,\quad t\geq 0,$$

which is a model with the "hereditary" wear term considered in [17] that satisfies the differential equation : $\partial_t w[p](x,t)=-\mu w[p](x,t)+\nu p(x,t)$. On the other hand, in the limit $\mu\to 0$, relation (6) reduces to

$$w[p](x,t)=\frac{\nu}{\Gamma(\alpha)}\int_0^t \frac{p(x,\tau)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau,\quad t\geq 0, \tag{8}$$

which, for $\alpha\in(0,1)$, solves the fractional order differential equation [10] :

$$D_t^\alpha w[p](x,t):=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{w[p](x,\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau=\nu p(x,t),\quad t\geq 0. \tag{9}$$

Note that since $w[p](x,0)=0$, the notions of Riemann-Liouville and Caputo fractional derivatives coincide for $\alpha\in(0,1)$. Relation (9) generalises the case $\alpha=1$ of the classical Archard's law [3] : $\partial_t w[p](x,t)=\nu p(x,t)$. The sliding punch problem with such a fractional derivative has been considered in [4]. Note that for arbitrary $\mu>0$, $\alpha\in(0,1)$, equation (6) can be written in a fractional differential form [13] :

$$D_t^\alpha w[p](x,t)=-\mu w[p](x,t)+\nu p(x,t).$$

2 Theoretical results

The model (1)–(2) has been rigorously analysed in [13, Sec. 4]. Following [14], we adapt here the general theory for a particular form of the kernel function (4).

To state the results, we first need to introduce some auxiliary quantities. For K in (4), consider

$$\mathcal{K}_2[\phi](x) := \int_{-a}^{a}\left[K(x-\xi) - \frac{1}{2a}\int_{-a}^{a}\left(K(\zeta - x) + K(\zeta-\xi)\right)\mathrm{d}\zeta\right]\phi(\xi)\,\mathrm{d}\xi,$$

which defines a positive compact self-adjoint operator on $L_0^2(-a,a)$, the space of square-integrable functions on $(-a,a)$ with vanishing mean. We denote its all (positive) eigenvalues arranged in decreasing order as $(\sigma_k)_{k\geq 1}$, and we assume that the corresponding eigenfunctions $(\phi_k)_{k\geq 1}$ are normalised to 1 in the $L_2(-a,a)$ norm. Furthermore, let us define the Mittag-Leffler function

$$E_\alpha(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad z\in\mathbb{C}, \quad \alpha>0. \tag{10}$$

2.1 Solvability and representation of the solution

Assume that $\mu \geq 0$, η, $\nu > 0$, $\alpha \in (0,2)$, $a \neq 2$, and $p(\cdot,0) \in L^2(-a,a)$ solves (5) with K given by (4) and $\delta(0)$ such that $\int_{-a}^{a} p(x,0)\,\mathrm{d}x = P_0$ for some $P_0 > 0$. Suppose $P(t)$ is as in (3) and $w[p]$ is defined in (6). Then, the unique solution $p \in C_b\left(\mathbb{R}_+; L^2(-a,a)\right)$ of integral equation (1) satisfying condition (2) is given by

$$p(x,t) = \frac{P(t)}{2a} + \sum_{k=1}^{\infty} d_k(t)\,\phi_k(x), \quad x\in(-a,a), \quad t\geq 0, \tag{11}$$

where

$$\begin{aligned} d_k(t) :=& d_k^0\left[1 + \frac{\nu}{\mu(\eta+\sigma_k)+\nu}\left(E_\alpha\left(-\left(\mu + \frac{\nu}{\eta+\sigma_k}\right)t^\alpha\right) - 1\right)\right] \\ &- \frac{l_k}{2a(\eta+\sigma_k)}\left(P(t) - P(0)\right) - \frac{\nu l_k}{2a(\eta+\sigma_k)^2}\left(\mu + \frac{\nu}{\eta+\sigma_k}\right)^{1/\alpha - 1} \\ &\times \int_0^t \mathcal{E}_\alpha\left(\left(\mu + \frac{\nu}{\eta+\sigma_k}\right)^{1/\alpha}(t-\tau)\right)\left[P(\tau) - P(0)\right]\mathrm{d}\tau, \quad k\geq 1, \end{aligned}$$

$$d_k^0 := \int_{-a}^{a} p(\xi,0)\,\phi_k(\xi)\,\mathrm{d}\xi, \qquad l_k := \frac{1}{2a}\int_{-a}^{a}\int_{-a}^{a} K(\zeta-\xi)\,\phi_k(\xi)\,\mathrm{d}\zeta\mathrm{d}\xi, \quad k\geq 1,$$

$\mathcal{E}_\alpha$, E_α are special functions defined in (7), (10), and $(\phi_k)_{k\geq 1} \subset L_0^2(-a,a)$, $(\sigma_k)_{k\geq 1} \subset \mathbb{R}_+$ are spectral parameters introduced in the previous paragraph.

2.2 Long-time behaviour of the solution : constant load case

First, let us consider the most commonly investigated case of a constant load, i.e. $P(t) \equiv P(0) = P_0$ for $t \geq 0$. The stationary pressure distribution can be computed to be

$$p_\infty(x) := \frac{P_0}{2a} + \sum_{k=1}^{\infty}\frac{\mu(\eta+\sigma_k)}{\mu(\eta+\sigma_k)+\nu} d_k^0 \phi_k(x). \tag{12}$$

Moreover, we can quantify how fast asymptotically the solution converges to this stationary distribution. Namely,

$$\|p(\cdot,t) - p_\infty\| = O\left(\exp\left(-\left(\mu + \frac{\nu}{\eta+\sigma_1}\right)t\right)\right), \quad t\gg 1, \quad \alpha = 1, \tag{13}$$

$$\|p(\cdot,t) - p_\infty\| = O\left(\frac{1}{t^\alpha}\right), \quad t\gg 1, \quad \alpha\in(0,1)\cup(1,2). \tag{14}$$

2.3 Long-time behaviour of the solution : oscillatory load case

Another common scenario is a load which is periodic in time. Focussing on harmonic regime (3), we expect that the solution would eventually we oscillatory with the same frequency. The following result confirms this expectation and also describes the stationary state, oscillation amplitude and phase, as well as the temporal rate of convergence.

There exists $\delta_p \in C\left(\mathbb{R}_+; L^2(-a,a)\right)$ with $\|\delta_p(\cdot,t)\|_{L^2(-a,a)} \to 0$ as $t \to +\infty$ such that the solution $p(x,t)$ to the model (1)–(2) can be written as

$$p(x,t) = \widetilde{p}_\infty(x) - W_0(x)\cos(\omega t - \psi(x)) + \delta_p(x,t), \tag{15}$$

where

$$\widetilde{p}_\infty(x) := \frac{1}{2a}(P_0 - P_\Delta) + \sum_{k=1}^{\infty}\left(d_k^0 \frac{\mu(\eta+\sigma_k)}{\mu(\eta+\sigma_k)+\nu} + \frac{P_\Delta \mu l_k}{2a[\mu(\eta+\sigma_k)+\nu]}\right)\phi_k(x), \tag{16}$$

$$W_0(x) := \left([W_1(x)]^2 + [W_2(x)]^2\right)^{1/2}, \qquad \psi(x) := \operatorname{sign}(W_2(x)) \arccos\frac{W_1(x)}{W_0(x)},$$

$$W_1(x) := -\frac{P_\Delta}{2a} + \frac{P_\Delta}{2a}\sum_{k=1}^{\infty}\frac{l_k}{\eta+\sigma_k}\left[\frac{\nu}{\eta+\sigma_k}\beta_k^{1-\alpha}C_k^{c,\omega} + 1\right]\phi_k(x),$$

$$W_2(x) := \frac{P_\Delta}{2a}\sum_{k=1}^{\infty}\frac{\nu l_k}{(\eta+\sigma_k)^2}\beta_k^{1-\alpha}C_k^{s,\omega}\phi_k(x),$$

$$C_k^{c,\omega} := \int_0^\infty \mathcal{E}_\alpha(\beta_k\tau)\cos(\omega\tau)\,d\tau, \quad C_k^{s,\omega} := \int_0^\infty \mathcal{E}_\alpha(\beta_k\tau)\sin(\omega\tau)\,d\tau,$$

$$\beta_k := \left(\mu + \frac{\nu}{\eta+\sigma_k}\right)^{1/\alpha}, \quad k \geq 1.$$

Moreover, we have

$$\|\delta_p(\cdot,t)\|_{L^2(-a,a)} = O\left(\exp\left(-\left(\mu + \frac{\nu}{\eta+\sigma_1}\right)t\right)\right), \quad t \gg 1, \quad \alpha = 1, \tag{17}$$

$$\|\delta_p(\cdot,t)\|_{L^2(-a,a)} = O\left(\frac{1}{t^\alpha}\right), \quad t \gg 1, \quad \alpha \in (0,1)\cup(1,2). \tag{18}$$

3 Numerical results

To illustrate results numerically, let us fix the following set of constants $P_0 = 6$, $C_K = 1.6$, $a = 1$, $\nu = 2$ and $\eta = 1$. Since we focus on the temporal behavior, instead of fixing initial punch profile $\Delta(x)$ and solving (5) for $p(x,0)$, for the sake of simplicity, we directly fix the initial pressure distribution $p(x,0) = \frac{2P_0}{a^2\pi}\left(a^2 - x^2\right)^{1/2}$. We evaluate the solution $p(x,t)$ according to (11) (with a series expansion truncated to 60 terms) following the preliminary computation of eigenfunctions $(\phi_k)_{k\geq 1}$ and eigenvalues $(\sigma_k)_{k\geq 1}$ by solving the integral equation $\mathcal{K}_2[\phi] = \sigma\phi$ using a collocation method. Note that this computational approach is validated (see [13, Sec. 6]) by comparison of the solution with a numerical solution that employs finite differences and quadrature rules.

First, we focus on the case of constant load $P(t) \equiv P_0$. In Figures 2 and 3, we study the dependence of the stationary pressure distribution $p_\infty(x)$ and the rate of the convergence on the model parameter μ and α. Note that the stationary profile $p_\infty(x)$ is impacted only by μ whereas α affects both quantitively and qualitatively the transient effects as is in agreement with (12)–(14).

Then, we proceed with the oscillarory load case where $P(t)$ is given by (3) with $P_\Delta = 0.5$ and $\omega = 1.5$. The situation in this case is slightly more complicated, since the stationary state $p_\infty(x,t) := \widetilde{p}_\infty(x) - W_0(x)\cos(\omega t - \psi(x))$ is not constant but oscillatory in time. To study the dependence of this stationary state on the model parameters, in Figure 4, we plot the envelope of $p_\infty(x,t)$ over the period $T = \frac{2\pi}{\omega} \simeq 4.19$ for several values of α and when $\mu = 1.2$ and $\mu = 0$. Furthermore, the dependence of the convergence rate on α and μ is illustrated in Figure 5 by plotting temporal behavior of the $L^\infty(-a,a)$ norm of the time-decaying term δ_p in (15).

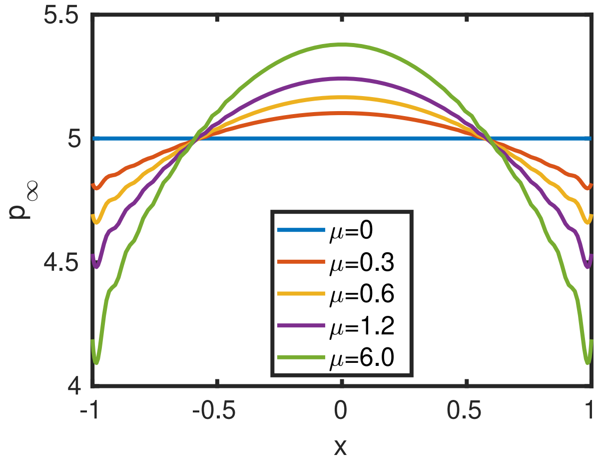

FIGURE 2 – Effect of the model parameter μ on the stationary state $p_\infty(x)$ (which is independent of α)

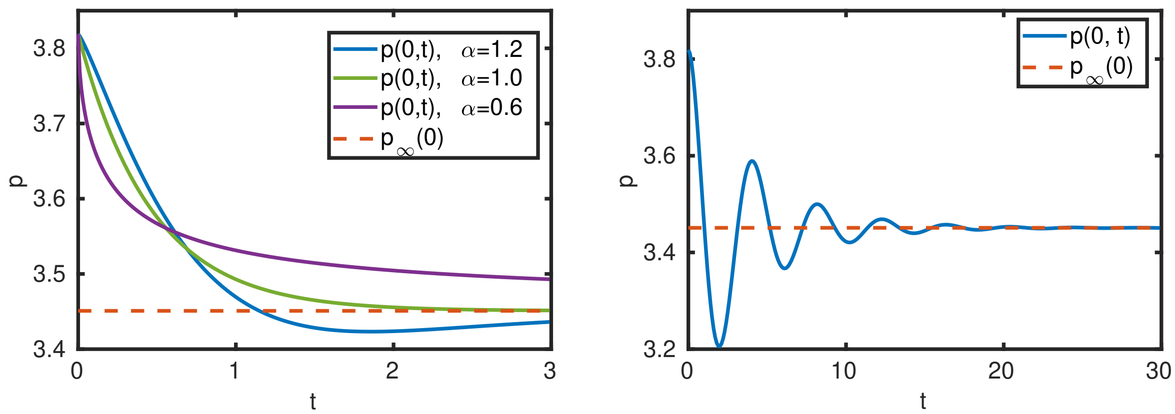

FIGURE 3 – Effect of the model parameter α on the convergence towards the stationary state p_∞ evaluated at $x = 0$ for fixed $\mu = 1.2$ and $\alpha \in \{0.6, 1.0, 1.2\}$ (left) and $\alpha = 1.8$ (right)

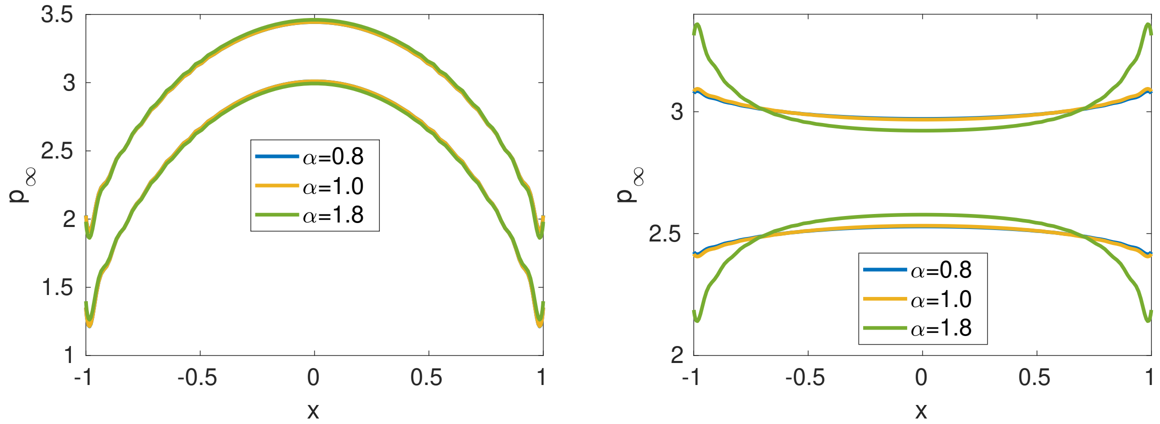

FIGURE 4 – Envelope of the stationary state $p_\infty(x,t)$ for 3 different values of α with $\mu = 1.2$ (left) and $\mu = 0$ (right)

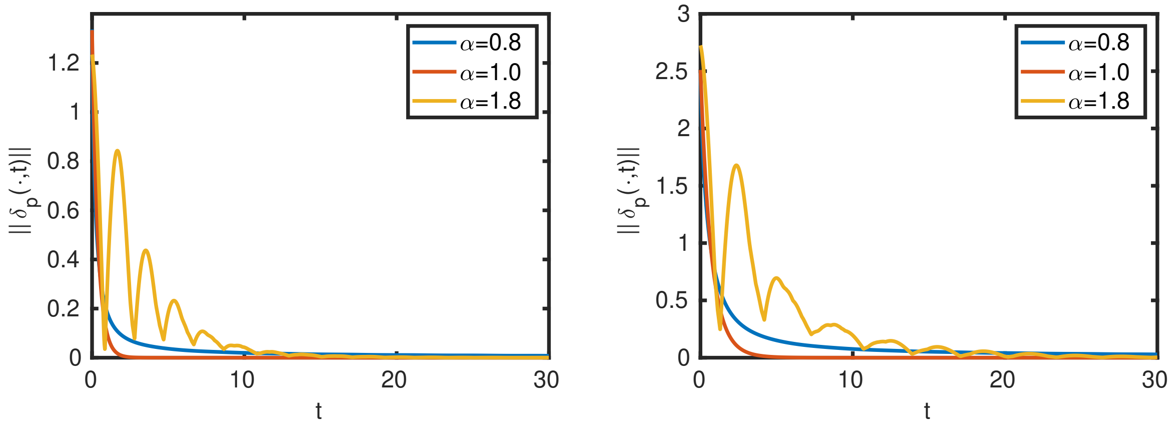

FIGURE 5 – Convergence of the solution to the stationary state for 3 different values of α with $\mu = 1.2$ (left) and $\mu = 0$ (right)

4 Discussion and conclusion

We considered a sliding punch problem with the wear process modelled through a non-local pressure-wear relation generalising the known ones in 2 different directions at once : we replaced regular time-derivative in the differential Archard's law with a fractional one and we took into account relaxation effects.

It is remarkable that the model is analytically solvable (in terms of some auxiliary spectral functions that can be precomputed) and its solution be analysed in a straigtforwad fashion for different loads (though we considered only two regimes : constant and oscillatory loads).

A practical advantage of such a model is that it admits more complicated behavior of the pressure distribution depending on the choice of model parameters. Namely, it allows different ways of temporal stabilisation towards a stationary state (generalising a classical exponential to an algebraic one which may even be non-monotone) and the stationary state itself need not be constant under a constant load. This may be useful for description of complex materials. Once the model parameters α and μ are fitted on the sliding punch problem, the deduced pressure-wear relation can be potentially employed for the same material pair and similar scenarios in more complicated mechanical problems.

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