Constrained optimization in classes of analytic functions with prescribed pointwise values

Laurent Baratchart, Juliette Leblond, Dmitry Ponomarev

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Abstract: We consider an overdetermined problem for Laplace equation on a disk with partial boundary data where additional pointwise data inside the disk have to be taken into account. After reformulation, this ill-posed problem reduces to a bounded extremal problem of best norm-constrained approximation of partial L^2 boundary data by traces of holomorphic functions which satisfy given pointwise interpolation conditions. The problem of best norm-constrained approximation of a given L^2 function on a subset of the circle by the trace of a H^2 function has been considered in [6]. In the present work, we extend such a formulation to the case where the additional interpolation conditions are imposed. We also obtain some new results that can be applied to the original problem: we carry out stability analysis and propose a novel method of evaluation of the approximation and blow-up rates of the solution in terms of a Lagrange parameter leading to a highly-efficient computational algorithm for solving the problem.

Key-words: Inverse boundary value problems, best norm-constrained approximation, holomorphic functions, Hardy spaces, pointwise interpolation.

RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE

2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex

Optimisation sous contraintes ponctuelles dans des classes de fonctions analytiques

Résumé : Nous considérons un problème inverse surdéterminé pour l'équation de Laplace dans un disque, avec des conditions de type Dirichlet-Neumann sur une partie de la frontière et des contraintes supplémentaires d'interpolation dans le disque. Après reformulation, ce problème est réduit à un problème de meilleure approximation quadratique sous contraintes, par les traces de fonctions holomorphes appartenant à l'espace de Hardy H^2, comme dans [6], et vérifiant des conditions d'interpolation dans le domaine. De plus, nous effectuons une analyse de la stabilité du problème face à des perturbations sur les données, et proposons une nouvelle méthode pour calculer certaines caractéristiques de la solution (erreur d'approximation, estimation de sa norme), en termes du paramètre de Lagrange intervenant dans l'algorithme.

Mots-clés : Problèmes inverses avec données frontière, meilleure approximation sous contrainte, fonctions holomorphes, espaces de Hardy, interpolation ponctuelle.

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1 Introduction

Many stationary physical problems are formulated in terms of reconstruction of function in a planar domain from partially available measurements on its boundary. Such problems may arise from 3-dimensional settings for which symmetry properties allow reformulation of the model in dimension 2, as it is often the case in Maxwell's equations for electro- or magnetostatics. It is a classical problem to find the function in the domain given values of the function itself (Dirichlet problem) or its normal derivative (Neumann problem) on the whole boundary. However, if the knowledge of Dirichlet or Neumann data is available only on a part of the boundary, the recovery problem is underdetermined and one needs to impose both conditions on the measurements on the accessible part of the boundary. One example of such a problem is recovering an electrostatic potential u satisfying conductivity equation in $\Omega \subset \mathbb{R}^2$ from knowledge of its values and those of the normal current $\sigma\partial_n u$ on a subset Γ of the boundary $\partial\Omega$ for a given conductivity coefficient σ in $\bar{\Omega}$:

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 & \text{in} \quad \Omega, \\ u = u_0, \quad \sigma\partial_n u = w_0 & \text{on} \quad \Gamma. \end{cases}$$

Often, only the behavior of the solution on $\partial\Omega\backslash\Gamma$ is of practical interest or, in case of a free boundary problem, one aims to find a position of this part of the boundary by imposing an additional condition there [3].

In the present work, we consider the prototypical case where the domain is the unit disk $\Omega = \mathbb{D}$, the conductivity coefficient is constant $\sigma \equiv 1$, and we assume appropriate regularity of the boundary data on $\Gamma \subset \mathbb{T}$ required for the existence of a unique weak $W^{1,2}(\Omega)$ solution:

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = u_0, \quad \partial_n u = w_0 & \text{on} \quad \Gamma \quad \text{with} \quad u_0 \in W^{1/2,2}(\Gamma), \;\; w_0 \in L^2(\Gamma). \end{cases} \tag{1.1}$$

Despite these simplifications, we note that since the Laplace operator is invariant under conformal transformations [1], results obtained for the unit disk can be readily extended to more general simply connected domains with smooth boundary. This is also true for non-constant conductivities σ since the same conformal invariance holds [10]. For non-constant conductivity equation, such a problem without additional pointwise data has been considered within the framework of generalized analytic functions [17] and beyond simply connected domains. In particular, the case of annular domain finds its application in a problem of recovery of plasma boundary in tokamaks [18].

The problem (1.1) happens to be ill-posed [24] as might be anticipated if one recalls the celebrated Hadamard's example which demonstrates lack of continuous dependence of solution on the boundary data if the boundary conditions assumed only on a part of the boundary. In the present formulation, as we are going to see, the both boundary conditions on Γ cannot be completely arbitrary functions as well: certain compatibility is needed in order for the solution to exist and be finite on $\partial\Omega\backslash\Gamma$. Therefore, one would like to find the admissible solution (that is bounded or, even more, sufficiently close to some *a priori* known data on $\partial\Omega\backslash\Gamma$) which is in best agreement with the given data u_0, w_0 on Γ. Put this way, the recovery issue is approximately recast as a well-posed constrained optimization problem, as we will show further.

In our approach, we use complex analytic tools to devise solution of the problem. Recall that if a function $g = u + iv$ is analytic (holomorphic), then u and v are real-valued harmonic functions satisfying the Cauchy-Riemann equations $\partial_n u = \partial_t v$, $\partial_t u = -\partial_n v$, where the partial derivatives are taken with respect to polar coordinates [27]. Applied to problem (1.1), the first equation suggests that knowing w_0, one can, up to an additive constant, recover v on Γ, and therefore both u_0 and w_0 define the trace on Γ of the analytic function g in Ω. However, the knowledge of an analytic function on a subset $\Gamma \subset \mathbb{T}$ of a positive measure defines the analytic function in the whole unit disk $\mathbb{D}$ [20, 31] and also its values on the unit circle $\mathbb{T}$ [21]. Of course, available data u_0, w_0 on Γ may not be compatible to yield the restriction of an analytic function onto Γ, and such instability phenomenon illustrates ill-posedness of the problem from the viewpoint of complex analysis.

As already mentioned, (1.1) may be recast as a well-posed bounded extremal problem in normed Hardy spaces of holomorphic functions defined by their boundary values. Different aspects of such problems were extensively considered in [5, 6, 8] and an algorithm for computation of the solution was proposed. In the present work, we would like to solve such an optimization problem incorporating additional available information from inside the domain. Taking advantage of the complex analytic approach allowing to make sense of pointwise values (unlike in Lebesgue spaces), we would like to extend previously obtained results to a situation where the solution needs to meet prescribed values at some points inside the disk. We characterize the solution in a way suitable for further practical implementation, obtain estimates of approximation rate and discrepancy growth, investigate the question of stability, illustrate numerically certain technical aspects, discuss the choice of auxiliary parameters and, based on a newly developed method of obtaining estimates on solution (which also applies to the problem without pointwise constraints), propose an improvement of the computational algorithm for solving the problem.

The paper is organized as follows. Section 2 provides an introduction to the theory of Hardy spaces which are essential functional spaces in the present approach. In Section 3, we formulate the problem, prove existence of a unique solution and give its useful characterization. Section 4 discusses the choice of interpolation function which is a technical tool to prescribe desired values inside the domain; we also provide an alternative form of the solution that turns out to be useful later. In Section 5, we obtain specific balance relations governing approximation rate on a given subset of the circle and discrepancy on its complement, which shed light on the quality of the solution depending on a choice of some auxiliary parameters. Also, at this point we introduce a novel series expansion method of evaluation of quantities governing solution quality. Section 6 introduces a closely related problem whose solution might be computationally cheaper in certain cases. We further look into sensitivity of the solution to perturbations of all input data in Section 7 raising the stability issue and providing technical estimates. We conclude with Section 8 by presenting numerical illustrations of certain properties of the solution, a short discussion of the choice of technical parameters and suggestion of a new efficient computational algorithm based on the results of the Section 5. Some concluding remarks are given in Section 9.

2 Background in theory of Hardy spaces

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$ with boundary $\mathbb{T}$.
Hardy spaces $H^p(\mathbb{D})$ can be defined as classes of holomorphic functions on the disk with finite norms

$$\|F\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| F\left(re^{i\theta}\right) \right|^p d\theta \right)^{1/p}, \quad 1 \le p < \infty,$$
$$\|F\|_{H^\infty} = \sup_{|z|<1} |F(z)| .$$

These are Banach spaces that enjoy plenty of interesting properties and they have been studied in detail over the years [15, 19, 21, 34]. In this section we give a brief introduction into the topic, yet trying to be as much self-contained as possible, adapting general material to our particular needs.
The key property of functions in Hardy spaces is their behavior on the boundary $\mathbb{T}$ of the disk. More precisely, boundary values of functions belonging to the Hardy space H^p are well-defined in the L^p sense

$$\lim_{r\nearrow 1} \left\| F\left(re^{i\theta}\right) - F\left(e^{i\theta}\right) \right\|_{L^p(\mathbb{T})} = 0, \quad 1 \le p < \infty, \tag{2.1}$$

as well as pointwise, for almost every $\theta \in [0, 2\pi]$:

$$\lim_{r\nearrow 1} F\left(re^{i\theta}\right) = F\left(e^{i\theta}\right). \tag{2.2}$$

It is the content of the Fatou's theorem (see, for instance, [21]) that the latter limit exists almost everywhere not only radially but also along any non-tangential path. Thanks to the Parseval's identity, the proof of (2.1) is especially simple when $p = 2$ (see [26, Th. 1.1.10]), the case that we will work with presently.
Given a boundary function $f \in L^p(\mathbb{T})$, $1 \le p \le \infty$ whose Fourier coefficients of negative index vanish

$$f_{-n} := \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{i\theta}\right) e^{in\theta} d\theta = 0, \quad n = 1, 2, \ldots, \tag{2.3}$$

(in this case, we say $f \in H^p(\mathbb{T})$), there exists $F \in H^p(\mathbb{D})$ such that $F\left(re^{i\theta}\right) \to f\left(e^{i\theta}\right)$ in L^p as $r \nearrow 1$, and it is defined by the Poisson representation formula, for $re^{i\theta} \in \mathbb{D}$:

$$F\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{it}\right) P_r(\theta - t)\, dt, \tag{2.4}$$

where we employed the Poisson kernel of $\mathbb{D}$:

$$P_r(\theta) := \frac{1 - r^2}{1 - 2r\cos\theta + r^2} = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}, \; 0 < r < 1, \quad \theta \in [0, 2\pi].$$

Note that the vanishing condition for the Fourier coefficients of negative order is equivalent to the requirement of the Poisson integral (2.4) to be analytic in $\mathbb{D}$. Indeed, since $f\left(e^{i\theta}\right) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}$, the right-hand side of (2.4) reads

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{it}\right) P_r(\theta - t)\, dt &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta} \sum_{n=-\infty}^{\infty} f_n \int_0^{2\pi} e^{i(n-k)t} dt = \sum_{n=-\infty}^{\infty} f_n r^{|n|} e^{in\theta} \\ &= f_0 + \sum_{n=1}^{\infty} \left(f_n z^n + f_{-n} \bar{z}^n \right), \end{aligned}$$

and hence, if we want this to define a holomorphic function through (2.4), we have to impose condition (2.3).
Because of the established isomorphism, we can identify the space $H^p = H^p(\mathbb{D})$ with $H^p(\mathbb{T}) \subset L^p(\mathbb{T})$ for $p \geq 1$ (the case $p = 1$ requires more sophisticated reasoning invoking F. & M. Riesz theorem [21]). It follows that H^p is a Banach space (as a closed subspace of $L^p(\mathbb{T})$ which is complete), and we have inclusions due to properties of Lebesgue spaces on bounded domains

$$H^\infty \subseteq H^s \subseteq H^p, \quad s \geq p \geq 1. \tag{2.5}$$

Summing up, we can abuse notation employing only one letter f, and write

$$\|f\|_{H^p} = \|f\|_{L^p(\mathbb{T})} \tag{2.6}$$

whenever $f \in L^p(\mathbb{T})$, $p \geq 1$, satisfies (2.3).
Moreover, in case $p = 2$, which we will focus on, the Parseval's identity provides an isometry between the Hardy space $H^2 = H^2(\mathbb{D})$ and the space $l_2(\mathbb{N}_0)$ of square-summable sequences [1]. Hence, H^2 is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2(\mathbb{T})} = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\theta}\right)\overline{g\left(e^{i\theta}\right)}\,d\theta = \sum_{k=0}^{\infty} f_k \bar{g}_k. \tag{2.7}$$

We will also repeatedly make use of the fact that H^∞ functions act as multipliers in H^p, that is, $H^\infty \cdot H^p \subset H^p$.
There is another useful property of Hardy classes to perform factorization: if $f \in H^p$ and $f(z_j) = 0$, $z_j \in \mathbb{D}$, $j = 1, \dots, N$, then $f = bg$ with $g \in H^p$ and the finite Blaschke product $b \in H^\infty$ defined as

$$b(z) = e^{i\phi_0}\prod_{j=1}^{N}\left(\frac{z - z_j}{1 - \bar{z}_j z}\right) \tag{2.8}$$

for some constant $\phi_0 \in [0, 2\pi]$. Possibility of such factorization comes from the observation that each factor of $b(z)$ is analytic in $\mathbb{D}$ and automorphic since

$$|z|^2 + |z_j|^2 - |z|^2|z_j|^2 = |z|^2\left(1 - |z_j|^2/2\right) + |z_j|^2\left(1 - |z|^2/2\right) \leq 1,$$

and thus

$$\left|\frac{z - z_j}{1 - \bar{z}_j z}\right|^2 = \frac{1 - 2\mathrm{Re}\left(\bar{z}_j z\right) + |z|^2 + |z_j|^2 - 1}{1 - 2\mathrm{Re}\left(\bar{z}_j z\right) + |z|^2|z_j|^2} \leq 1.$$

Additionally, this shows that

$$|b| \equiv 1, \quad z \in \mathbb{T}, \tag{2.9}$$

and hence $\|b\|_{H^\infty} = 1$.
We let $\bar{H}_0^2$ denote the orthogonal complement of H^2 in $L^2(\mathbb{T})$, so that $L^2 = H^2 \oplus \bar{H}_0^2$. Recalling characterization (2.3) of H^2 functions, we can view $\bar{H}_0^2$ as the space of functions whose expansions have non-vanishing Fourier coefficients of only negative index, and hence it characterizes $L^2(\mathbb{T})$ functions which are holomorphic in $\mathbb{C}\backslash\bar{\mathbb{D}}$ and decay to zero at infinity.
Similarly, we can introduce the orthogonal complement to bH^2 in $L^2(\mathbb{T})$ with b as in (2.8) so that $L^2 = bH^2 \oplus \left(bH^2\right)^\perp$ which in its turn decomposes into a direct sum as $\left(bH^2\right)^\perp = \bar{H}_0^2 \oplus \left(bH^2\right)^{\perp_{H^2}}$ with $\left(bH^2\right)^{\perp_{H^2}} \subset H^2$ denoting the orthogonal complement to bH^2 in H^2; it is not empty if

[1] Here and onwards, we stick to the convention: $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, $\mathbb{N}_+ := \{1, 2, 3, \dots\}$.

$b \not\equiv$ const, whence the proper inclusion $bH^2 \subset H^2$ holds. Moreover, making use of the Cauchy integral formula, it is not difficult to show that

$$\left(bH^2\right)^{\perp_{H^2}} := \left(bH^2\right)^{\perp} \ominus \bar{H}_0^2 = \frac{P_{N-1}(z)}{\prod_{j=1}^{N}(1-\bar{z}_j z)},$$

where $P_{N-1}(z)$ is the space of polynomials of degree at most $N-1$ in z.

Given $J \subset \mathbb{T}$, let us introduce the Toeplitz operator ϕ with symbol χ_J (the indicator function of J), defined by:

$$\begin{array}{rcl} H^2 & \to & H^2 \\ F & \mapsto & \phi(F) = P_+(\chi_J F), \end{array} \tag{2.10}$$

where we let P_+ denote the orthogonal projection from $L^2(\mathbb{T})$ onto H^2 (that might be realized by setting Fourier coefficients of negative index to zero or convolving the function with the Cauchy kernel). Similarly, $P_- := I - P_+$ defines the orthogonal projection onto $\bar{H}_0^2$.
We also notice that the map $L^2(\mathbb{T}) \to bH^2 : F \mapsto bP_+\left(\bar{b}F\right)$ is the orthogonal projection onto bH^2. Indeed, taking into account (2.9), for any $u \in L^2(\mathbb{T})$, $v \in H^2$,

$$\left\langle u - bP_+\left(\bar{b}u\right), bv\right\rangle_{L^2(\mathbb{T})} = \langle u, bv\rangle_{L^2(\mathbb{T})} - \left\langle P_+\left(\bar{b}u\right), \bar{b}bv\right\rangle_{L^2(\mathbb{T})} = 0.$$

Any function in H^p, $p \geq 1$, being analytic and sufficiently regular on $\mathbb{T}$, admits integral representation in terms of its boundary values and thus is uniquely determined by means of the Cauchy formula. However, it is also possible to recover a function f holomorphic in $\mathbb{D}$ from its values on a subset of the boundary $I \subset \mathbb{T}$ using so-called Carleman's formulas [4, 20]. Write $\mathbb{T} = I \cup J$ with I and J being Lebesgue measurable sets.

Proposition 1. *Assume $|I| > 0$ and let $\Phi \in H^\infty$ be any function such that $|\Phi| > 1$ in $\mathbb{D}$ and $|\Phi| = 1$ on J. Then, $f \in H^p$, $p \geq 1$ can be represented from $f|_I$ as*

$$f(z) = \frac{1}{2\pi i} \lim_{\alpha \to \infty} \int_I \frac{f(\xi)}{\xi - z} \left[\frac{\Phi(\xi)}{\Phi(z)}\right]^{\alpha} d\xi, \tag{2.11}$$

where the convergence is uniform on compact subsets of $\mathbb{D}$.

Proof. Since $\Phi \in H^\infty$ and $f \in H^p \subseteq H^1$, it is clear that $f(z)[\Phi(z)]^\alpha \in H^1$, and so the Cauchy formula applies to $f(z)[\Phi(z)]^\alpha = f(z)\exp[\alpha \log \Phi(z)]$ for any $\alpha > 0$

$$f(z)[\Phi(z)]^\alpha = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\xi)[\Phi(\xi)]^\alpha}{\xi - z} d\xi$$

$$\Rightarrow \quad f(z) = \frac{1}{2\pi i}\left(\int_I + \int_J\right) \frac{f(\xi)}{\xi - z}\left[\frac{\Phi(\xi)}{\Phi(z)}\right]^{\alpha} d\xi.$$

Since the second integral vanishes in absolute value as $\alpha \nearrow \infty$ for any $z \in \mathbb{D}$ (by the choice of Φ), we have (2.11). $\square$

The integral representation (2.11) implies the following uniqueness result (see also e.g. [34, Th. 17.18], for a different argument based on the factorization which shows that $\log|f| \in L^1(\mathbb{T})$ whenever $f \in H^p$).

Corollary 1. *Functions in H^1 are uniquely determined by their boundary values on $I \subset \mathbb{T}$ as soon as $|I| > 0$.*

It follows that if two H^p functions agree on a subset of $\mathbb{T}$ with non-zero Lebesgue measure, then they must coincide everywhere in $\mathbb{D}$. This complements the identity theorem for holomorphic functions [1] claiming that zero set of an analytic function cannot have an accumulation point inside the domain of analyticity which particularly implies that two functions coinciding in a neighbourhood of a point of analyticity are necessarily equal in the whole domain of analyticity.

Remark 1. Using the isometry $H^2 \to \bar{H}^2_0$:

$$f(z) \mapsto \frac{1}{z}\overline{f\left(\frac{1}{\bar{z}}\right)}, \ z \in \mathbb{D}$$

(which is clear from the Fourier expansion on the boundary), we check that Proposition 1 and Corollary 1 also apply to functions in $\bar{H}^2_0$.

Remark 2. The auxiliary function Φ termed as "quenching" function can be chosen as follows. Let u be a Poisson integral of a positive function vanishing on J (for instance, the characteristic function χ_I) and v its harmonic conjugate that can be recovered (up to an additive constant) at $z = re^{i\theta}$, $r < 1$ by convolving u on $\mathbb{T}$ (using normalized Lebesgue measure $d\sigma = \frac{1}{2\pi}d\theta$) with the conjugate Poisson kernel $\operatorname{Im}\left(\frac{1+re^{it}}{1-re^{it}}\right)$, $t \in [0, 2\pi]$, see [21] for details. Then, clearly, $\Phi = \exp(u + iv)$ is analytic in $\mathbb{D}$ and satisfies the required conditions. More precisely, combining recovered v with the Poisson representation formula for u, we conclude that convolution of boundary values of u with the Schwarz kernel $\frac{1+re^{it}}{1-re^{it}}$, $t \in [0, 2\pi]$ defines (up to an additive constant) the analytic function $u(z) + iv(z)$ for $z = re^{i\theta} \in \mathbb{D}$. An explicit quenching function constructed in such a way will be given in Section 3 by (3.30).

Remark 3. A similar result was also obtained and discussed in [31], see also [4, 8, 23].

As a consequence of Remark 1, we derive a useful tool in form of

Proposition 2. *The Toeplitz operator ϕ is an injection on H^2.*

Proof. By the orthogonal decomposition $L^2 = H^2 \oplus \bar{H}^2_0$, we have $\chi_J g = P_+(\chi_J g) + P_-(\chi_J g)$. Now, if $P_+(\chi_J g) = 0$, then $\chi_J g$ is a $\bar{H}^2_0$ function vanishing on I and hence, by Remark 1, must be identically zero. □

The last result for Hardy spaces that we are going to employ is the density of traces [6, 8].

Proposition 3. *Let $J \subset \mathbb{T}$ be a subset of non-full measure, that is $|I| = |\mathbb{T} \backslash J| > 0$. Then, the restriction $H^p|_J := (trH^p)|_J$ is dense in $L^p(J)$, $1 \leq p < \infty$.*

Proof. In the particular case $p = 2$ (other values of p are treated in [6]), we prove the claim by contradiction. Assume that there is non-zero $f \in L^2(J)$ orthogonal to $H^2\big|_J$, then, extending it by zero on I, we denote the extended function as $\tilde{f}$. We thus have $\left\langle \tilde{f}, g\right\rangle_{L^2(\mathbb{T})} = 0$ for all $g \in H^2$ which implies $\tilde{f} \in \bar{H}^2_0$ and hence, by Remark 1, $f \equiv 0$.

□

Remark 4. From the proof and Remark 1, we see that the same density result holds if one replaces H^2 with $\bar{H}^2_0$.

There is a counterpart of Propositon 3 that also characterizes boundary traces of H^p spaces.

Proposition 4. *Assume* $|I| > 0$, $f \in L^p(I)$, $1 \le p \le \infty$. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of H^p functions such that $\lim_{n\to\infty} \|f - g_n\|_{L^p(I)} = 0$. Then, $\|g_n\|_{L^p(J)} \to \infty$ as $n \to \infty$ unless f is the trace of a H^p function.

Proof. Consider the case $1 < p < \infty$; for the cases $p = 1$ and $p = \infty$ we refer to [6] and [8], respectively. We argue by contradiction: assume that f is not the trace on I of some H^p function, but $\lim_{n\to\infty} \|g_n\|_{L^p(J)} < \infty$. Then, by hypothesis, the sequence $\{g_n\}_{n=1}^{\infty}$ is bounded not only in $L^p(J)$ but also in H^p. Since H^p is reflexive (as any $L^p(\mathbb{T})$ is for $1 < p < \infty$), it follows from the Banach-Alaoglu theorem (or see [25, Ch. 10, Th. 7]) that the closed unit ball in H^p is weakly compact, therefore, we can extract a subsequence $\{g_{n_k}\}$ that converges weakly in H^p: $g_{n_k} \rightharpoonup g$ for some $g \in H^p$. However, since $g_n \to f$ in $L^p(I)$, we must have $f = g|_I$, a contradiction. □

Remark 5. When $|J| = 0$, the existence of a H^p sequence $\{g_n\}_{n=1}^{\infty}$ approximating $f \in L^p(I)$ in $L^p(I)$ norm, means that f actually belongs to H^p (which is a closed subspace of $L^p(\mathbb{T}) = L^p(I)$).

3 An extremal problem and its solution

We consider the problem of finding a H^2 function which takes prescribed values $\{\omega_j\}_{j=1}^N \in \mathbb{C}$ at interior points $\{z_j\}_{j=1}^N \in \mathbb{D}$ which approximates best a given $L^2(I)$ function on a subset of the boundary $I \subset \mathbb{T}$ while remaining close enough to another $L^2(J)$ function on the complementary part $J \subset \mathbb{T}$.
We proceed with a technical formulation of this problem. Assuming given interpolation values at distinct interior points $\{z_j\}_{j=1}^N \in \mathbb{D}$, we let $\psi \in H^2$ be some fixed function satisfying the interpolation conditions

$$\psi(z_j) = \omega_j \in \mathbb{C}, \quad j = 1, \dots, N. \tag{3.1}$$

Then, any interpolating function in H^2 fulfiling these conditions can be written as $\tilde{g} = \psi + bg$ for arbitrary $g \in H^2$ with $b \in H^\infty$ the finite Blaschke product defined in (2.8).
As before, let $\mathbb{T} = I \cup J$ with both I and J being of non-zero Lebesgue measure. For the sake of simplicity, we write $f = f|_I \vee f|_J$ to mean a function defined on the whole $\mathbb{T}$ through its values given on I and J.
For $h \in L^2(J)$, $M \geq 0$, let us introduce the following functional spaces

$$\mathcal{A}^{\psi,b} := \left\{\tilde{g} \in H^2 : \tilde{g} = \psi + bg,\, g \in H^2\right\}, \tag{3.2}$$

$$\mathcal{B}_{M,h}^{\psi,b} := \left\{g \in H^2 : \|\psi + bg - h\|_{L^2(J)} \leq M\right\}, \tag{3.3}$$

$$\mathcal{C}_{M,h}^{\psi,b} := \left\{f \in L^2(I) : f = \psi|_I + b\, g|_I \,,\, g \in \mathcal{B}_{M,h}^{\psi,b}\right\}. \tag{3.4}$$

We then have inclusions $\mathcal{C}_{M,h}^{\psi,b} \subseteq \mathcal{A}^{\psi,b}\big|_I \subseteq H^2\big|_I \subset L^2(I)$ and $\mathcal{C}_{M,h}^{\psi,b} = \left(\psi + b\mathcal{B}_{M,h}^{\psi,b}\right)\Big|_I \neq \emptyset$ since $\mathcal{B}_{M,h}^{\psi,b} \neq \emptyset$ for any given $h \in L^2(J)$ and $M > 0$ as follows from Proposition 3.
Now the framework is set to allow us to pose the problem in precise terms.
Given $f \in L^2(I)$, our goal will be to find a solution to the following bounded extremal problem

$$\min_{g \in \mathcal{B}_{M,h}^{\psi,b}} \|\psi + bg - f\|_{L^2(I)}. \tag{3.5}$$

As it was briefly mentioned at the beginning, the motivation for such a formulation is to look for

$$\tilde{g}_0 := \psi + bg_0 \in \mathcal{A}^{\psi,b} \quad \text{such that} \quad g_0 = \arg \min_{g \in \mathcal{B}_{M,h}^{\psi,b}} \big\|\underbrace{\psi + bg}_{=\tilde{g}} - f\big\|_{L^2(I)}, \tag{3.6}$$

i.e. the best H^2-approximant to f on I which fulfils interpolation conditions (3.1) and is not too far from the reference h on J: $\|\tilde{g}_0 - h\|_{L^2(J)} \leq M$. In view of Proposition 4, the L^2-constraint on J is crucial whenever $f \notin \mathcal{A}^{\psi,b}\big|_I$ (which is always the case when known data are recovered from physical measurements necessarily subject to noise). In other words, we assume that

$$g|_I \neq \bar{b}(f - \psi), \tag{3.7}$$

i.e. there is no $\tilde{g} = \psi + bg \in H^2$ whose trace on I is exactly the given function $f \in L^2(I)$, and at the same time remains within the L^2-distance M from h on J. This motivates the choice (3.3) for the space of admissible solutions $\mathcal{B}_{M,h}^{\psi,b}$.
Existence and uniqueness of solution to (3.5) can be reduced to what has been proved in a general setting in [6]. Here we present a slightly different proof.

Theorem 1. *For any $f \in L^2(I)$, $h \in L^2(J)$, $\psi \in H^2$, $M \geq 0$ and $b \in H^\infty$ defined as (2.8), there exists a unique solution to the bounded extremal problem (3.5).*

Proof. By the existence of a best approximation projection onto a non-empty closed convex subset of a Hilbert space (see, for instance, [13, Th. 3.10.2]), it is required to show that the space of restrictions $\mathcal{B}_{M,h}^{\psi,b}\Big|_I$ is a closed convex subset of $L^2(I)$. Convexity is a direct consequence of the triangle inequality:

$$\|\alpha(bg_1 + \psi - h) + (1-\alpha)(bg_2 + \psi - h)\|_{L^2(J)} \leq \alpha M + (1-\alpha) M = M$$

for any g_1, $g_2 \in \mathcal{B}_{M,h}^{\psi,b}$ and $\alpha \in [0,1]$.

We will now show the closedness property. Let $\{g_n\}_{n=1}^\infty$ be a sequence of $\mathcal{B}_{M,h}^{\psi,b}$ functions which converges in $L^2(I)$ to some g: $\|g - g_n\|_{L^2(I)} \to 0$ as $n \to \infty$. We need to prove that $g \in \mathcal{B}_{M,h}^{\psi,b}$. We note that $g \in H^2\big|_I$, since otherwise, by Proposition 4, $\|g_n\|_{L^2(J)} \to \infty$ as $n \to \infty$, which would contradict the fact that $g_n \in \mathcal{B}_{M,h}^{\psi,b}$ starting with some n. Therefore, $\psi + bg \in H^2$ and $\langle \psi + bg, \xi \rangle_{L^2(\mathbb{T})} = 0$ for any $\xi \in \bar{H}_0^2$, which implies that

$$\langle \psi + bg, \xi \rangle_{L^2(I)} = \langle (\psi + bg) \vee 0, \xi \rangle_{L^2(\mathbb{T})} = -\langle 0 \vee (\psi + bg), \xi \rangle_{L^2(\mathbb{T})} = -\langle \psi + bg, \xi \rangle_{L^2(J)}.$$

From here, using the same identity for $\psi + bg_n$, we obtain

$$\begin{aligned}\langle \psi + bg - h, \xi \rangle_{L^2(J)} &= -\langle \psi + bg, \xi \rangle_{L^2(I)} - \langle h, \xi \rangle_{L^2(J)} = -\lim_{n\to\infty} \langle \psi + bg_n, \xi \rangle_{L^2(I)} - \langle h, \xi \rangle_{L^2(J)} \\ &= \lim_{n\to\infty} \langle \psi + bg_n, \xi \rangle_{L^2(J)} - \langle h, \xi \rangle_{L^2(J)}.\end{aligned}$$

Since $g_n \in \mathcal{B}_{M,h}^{\psi,b}$ for all n, the Cauchy-Schwarz inequality gives

$$\left|\langle \psi + bg - h, \xi \rangle_{L^2(J)}\right| = \lim_{n\to\infty} \left|\langle \psi + bg_n - h, \xi \rangle_{L^2(J)}\right| \leq M \|\xi\|_{L^2(J)}$$

for any $\xi \in \bar{H}_0^2\big|_J$. The final result is now furnished by employing density of $\bar{H}_0^2\big|_J$ in $L^2(J)$ (Proposition 3 and Remark 4) and the dual characterization of $L^2(J)$ norm:

$$\|\psi + bg - h\|_{L^2(J)} = \sup_{\substack{\xi \in L^2(J) \\ \|\xi\|_{L^2(J)} \leq 1}} \left|\langle \psi + bg - h, \xi \rangle_{L^2(J)}\right| = \sup_{\substack{\xi \in \bar{H}_0^2 \\ \|\xi\|_{L^2(J)} \leq 1}} \left|\langle \psi + bg - h, \xi \rangle_{L^2(J)}\right| \leq M.$$

□

A key property of the solution is that the constraint in (3.3) is necessarily saturated unless $f \in \mathcal{A}^{\psi,b}\big|_I$.

Lemma 1. *If $f \notin \mathcal{A}^{\psi,b}\big|_I$ and $g \in \mathcal{B}_{M,h}^{\psi,b}$ solves (3.5), then $\|\psi + bg - h\|_{L^2(J)} = M$.*

Proof. To show this, suppose the opposite, i.e. there is $g_0 \in H^2$ solving (3.5) for which we have

$$\|\psi + bg_0 - h\|_{L^2(J)} < M.$$

The last condition means that g_0 is in interior of $\mathcal{B}_{M,h}^{\psi,b}$, and hence we can define $g^\star := g_0 + \epsilon\delta_g \in \mathcal{B}_{M,h}^{\psi,b}$ for sufficiently small $\epsilon > 0$ and $\delta_g \in H^2$, $\|\delta_g\|_{H^2} = 1$ such that $\mathrm{Re}\,\langle b\delta_g, \psi + bg_0 - f \rangle_{L^2(I)} < 0$, where the equality case is eliminated by (3.7). By the smallness of ϵ, the quadratic term is negligible, and thus we have

$$\begin{aligned}\left\|\psi+bg^{\star}-f\right\|_{L^{2}(I)}^{2} &= \left\|\psi+bg_{0}-f\right\|_{L^{2}(I)}^{2}+2\epsilon\operatorname{Re}\left\langle b\delta_{g},\psi+bg_{0}-f\right\rangle_{L^{2}(I)}+\epsilon^{2}\left\|\delta_{g}\right\|_{L^{2}(I)}^{2}\\ &< \left\|\psi+bg_{0}-f\right\|_{L^{2}(I)}^{2},\end{aligned}$$

which contradicts the minimality of g_0. □

As an immediate consequence of saturation of the constraint, we obtain

Corollary 2. *The requirement $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}\big|_I$ implies that the formulation of the problem should be restricted to the case $M > 0$.*

Proof. If $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}\big|_I$ and $M = 0$, the Lemma entails that $h \in \mathcal{A}^{\psi,b}\big|_J$. Then, $h = \psi + bg$ for some $g \in H^2$ and its extension to the whole $\mathbb{D}$ (given, for instance, by Proposition 1) uniquely determines $\tilde{g} = h$ without resorting to solution of the bounded extremal problem (3.5), hence independently of f. □

Having established that equality holds in (3.3), we approach (3.5) as a constrained optimization problem following a standard idea of Lagrange multipliers (e.g. [37]) and claim that for a solution g to (3.5) and for some $\lambda \in \mathbb{R}$, we must necessarily have

$$\left\langle\delta_{\tilde{g}},(\tilde{g}-f)\vee\lambda(\tilde{g}-h)\right\rangle_{L^{2}(\mathbb{T})}=0 \tag{3.8}$$

for any $\delta_{\tilde{g}} \in bH^2$ (recall that $\tilde{g} = \psi + bg$ and $\delta_{\tilde{g}} = b\delta_g$ for $\delta_g \in H^2$) which is a condition of tangency of level lines of the minimizing objective functional and the constraint functional. The condition (3.8) can be shown by the same variational argument as in the proof of Lemma 1, it must hold true, otherwise we would be able to improve the minimum while still remaining in the admissible set. This motivates us to search for $g \in H^2$ such that, for $\lambda \in \mathbb{R}$,

$$\left[(\psi+bg-f)\vee\lambda(\psi+bg-h)\right]\in\left(bH^{2}\right)^{\perp} \tag{3.9}$$

which is equivalent to

$$P_{+}\left[\bar{b}(\psi+bg-f)\vee\lambda\bar{b}(\psi+bg-h)\right]=0. \tag{3.10}$$

Theorem 2. *If $f \notin \mathcal{A}^{\psi,b}\big|_I$, the solution to the bounded extremal problem (3.5) is given by*

$$g_{0}=(1+\mu\phi)^{-1}P_{+}\left(\bar{b}(f-\psi)\vee(1+\mu)\bar{b}(h-\psi)\right), \tag{3.11}$$

where the parameter $\mu > -1$ is uniquely chosen such that $\|\psi + bg_0 - h\|_{L^2(J)} = M$.

The proof of Theorem 2 goes in three steps.

3.1 Solution for the case $h = 0$

For simplicity, we first assume $h = 0$. Then, the equation (3.10) can be elaborated as follows

$$P_{+}\left(\bar{b}(\psi+bg)\right)+(\lambda-1)P_{+}\left(0\vee\bar{b}(\psi+bg)\right)=P_{+}\left(\bar{b}f\vee0\right),$$

$$g+P_{+}\left(\bar{b}\psi\right)+(\lambda-1)P_{+}\left(0\vee\bar{b}\psi\right)+(\lambda-1)\phi g=P_{+}\left(\bar{b}f\vee0\right),$$

$$(1+\mu\phi)\, g = -P_+ \left(\bar{b}\,(\psi - f) \vee (1+\mu)\,\bar{b}\psi\right), \tag{3.12}$$

where we introduced the parameter $\mu := \lambda - 1 \in \mathbb{R}$.
The Toeplitz operator ϕ, defined as (2.10), is self-adjoint and, as it can be shown (see the Hartman-Wintner theorem in Appendix), its spectrum is

$$\sigma(\phi) = [\text{ess inf } \chi_J, \text{ ess sup } \chi_J] = [0,1], \tag{3.13}$$

hence $\|\phi\| \leq 1$ and the operator $(1+\mu\phi)$ is invertible on H^2 for $\mu > -1$ allowing to claim that

$$g = -(1+\mu\phi)^{-1} P_+ \left(\bar{b}\,(\psi - f) \vee (1+\mu)\,\bar{b}\psi\right). \tag{3.14}$$

This generalizes the result of [5] to the case when solution needs to meet pointwise interpolation conditions.

3.2 Solution for the case $h \neq 0,\ h \in H^2\big|_J$

Now, let $h \neq 0$, but assume it to be the restriction to J of some H^2 function.
We write $f = \varrho + \kappa|_I$ for $\kappa \in H^2$ such that $\kappa|_J = h$. Then, the solution to (3.5) is

$$g_0 = \arg \min_{g \in \mathcal{B}^{\psi,b}_{M,h}} \|\psi + bg - f\|_{L^2(I)} = \arg \min_{g \in \tilde{\mathcal{B}}_{M,0}} \left\|\tilde{\psi} + bg - \varrho\right\|_{L^2(I)},$$

where $\tilde{\psi} := \psi - \kappa$ and

$$\tilde{\mathcal{B}}_{M,0} := \left\{ g \in H^2 : \left\|\tilde{\psi} + bg\right\|_{L^2(J)} \leq M \right\}.$$

It is easy to see that, due to $\kappa|_J = h$, we have $\tilde{\mathcal{B}}_{M,0} = \mathcal{B}^{\psi,b}_{M,h}$. Therefore, the already obtained results (3.12), (3.14) apply to yield

$$\begin{aligned}(1+\mu\phi)\, g_0 &= -P_+\left(\bar{b}\left(\tilde{\psi} - \varrho\right) \vee (1+\mu)\,\bar{b}\tilde{\psi}\right) \\ &= -P_+\left(\bar{b}\,(\psi - \kappa - \varrho) \vee (1+\mu)\,\bar{b}\,(\psi - \kappa)\right) \\ &= P_+\left(\bar{b}\,(f - \psi) \vee (1+\mu)\,\bar{b}\,(h - \psi)\right),\end{aligned} \tag{3.15}$$

from where (3.11) follows.

3.3 Solution for the case $h \neq 0,\ h \in L^2(J)$

Here we assume $h \notin H^2\big|_J$ but only $h \in L^2(J)$. The result follows from the previous step by density of $H^2\big|_J$ in $L^2(J)$ along the line of reasoning similar to [6].
More precisely, by density (Proposition 3), for a given $h \in L^2(J)$, we have existence of a sequence $\{h_n\}_{n=1}^{\infty} \subset H^2\big|_J$ such that $h_n \underset{n\to\infty}{\to} h$ in $L^2(J)$. This generates a sequence of solutions

$$g_n = \arg \min_{g \in \mathcal{B}_{M,h_n}} \|\psi + bg - f\|_{L^2(I)}, \quad n \in \mathbb{N}_+, \tag{3.16}$$

satisfying

$$(1+\mu_n\phi)\, g_n = P_+\left(\bar{b}\,(f - \psi) \vee (1+\mu_n)\,\bar{b}\,(h_n - \psi)\right) \tag{3.17}$$

for $\mu_n > -1$ chosen such that $\|\psi + bg_n - h_n\|_{L^2(J)} = M$.
Since $\{g_n\}_{n=1}^{\infty}$ is bounded in H^2 (by definition of the solution space $\mathcal{B}^{\psi,b}_{M,h_n}$), and due to the

Hilbertian setting, up to extraction of a subsequence, it converges weakly in $L^2(\mathbb{T})$ norm to some element in H^2

$$g_n \underset{n\to\infty}{\rightharpoonup} \gamma \in H^2. \tag{3.18}$$

We will first show that $\mu_n \to \mu$ as $n \to \infty$. Then, since all $(1+\mu\phi)$ and $(1+\mu_n\phi)$ are self-adjoint, we have, for any $\xi \in H^2$,

$$\langle (1+\mu_n\phi)\, g_n, \xi\rangle_{L^2(\mathbb{T})} = \langle g_n, (1+\mu_n\phi)\,\xi\rangle_{L^2(\mathbb{T})} \underset{n\to\infty}{\to} \langle \gamma, (1+\mu\phi)\,\xi\rangle_{L^2(\mathbb{T})} = \langle (1+\mu\phi)\,\gamma, \xi\rangle_{L^2(\mathbb{T})},$$

and thus $(1+\mu_n\phi)\, g_n \underset{n\to\infty}{\rightharpoonup} (1+\mu\phi)\,\gamma$. Combining this with the convergence

$$P_+\left(\bar{b}\,(f-\psi) \vee (1+\mu_n)\,\bar{b}\,(h_n-\psi)\right) \underset{n\to\infty}{\to} P_+\left(\bar{b}\,(f-\psi) \vee (1+\mu)\,\bar{b}\,(h-\psi)\right)$$

in $L^2(\mathbb{T})$, equation (3.17) suggests that the weak limit γ in (3.18) is a solution to (3.5). It will remain to check that $\gamma \in \mathcal{B}^{\psi,b}_{M,h}$ and is indeed a minimizer of the cost functional (3.5).

Claim 1. For μ_n in (3.17), we have

$$\lim_{n\to\infty} \mu_n =: \mu \in (-1, \infty)\,. \tag{3.19}$$

Proof. We prove this statement by contradiction. Because of the relation (3.10), for any $\xi \in H^2$, we have

$$\left\langle \bar{b}\,(f-\psi) - g_n, \xi\right\rangle_{L^2(I)} = (1+\mu_n)\left\langle g_n - \bar{b}\,(h_n-\psi)\,,\xi\right\rangle_{L^2(J)}. \tag{3.20}$$

We note that the weak convergence (3.18) in H^2 implies the weak convergence $g_n \rightharpoonup \gamma$ in $L^2(J)$ as $n \to \infty$ since for a given $\eta \in L^2(J)$, we can take $\xi = P_+(0 \vee \eta) \in H^2$ in the definition $\lim\limits_{n\to\infty} \langle g_n, \xi\rangle_{L^2(\mathbb{T})} = \langle \gamma, \xi\rangle_{L^2(\mathbb{T})}$.

Assume first that $\mu_n \underset{n\to\infty}{\to} \infty$. Then, since the left-hand side of (3.20) remains bounded as $n\to\infty$, we necessarily must have

$$\lim_{n\to\infty} \left\langle g_n - \bar{b}\,(h_n-\psi)\,,\xi\right\rangle_{L^2(J)} = 0.$$

Since $h_n \to h$ in $L^2(J)$ strongly, this implies that $\gamma = \bar{b}\,(h-\psi) \in H^2\big|_J$ contrary to the initial assumption of the section that $h \notin H^2\big|_J$.

Next, we consider another possibility, namely that the limit $\lim\limits_{n\to\infty}\mu_n$ does not exist. Then, there are at least two infinite sequences $\{n_{k_1}\}$, $\{n_{k_2}\}$ such that

$$\lim_{k_1\to\infty} \mu_{n_{k_1}} =: \mu^{(1)} \neq \mu^{(2)} := \lim_{k_2\to\infty} \mu_{n_{k_2}}.$$

Since the left-hand side of (3.20) is independent of μ_n and both limits $\mu^{(1)}$, $\mu^{(2)}$ exist and finite, we have

$$\lim_{k_1\to\infty}\left(1+\mu_{n_{k_1}}\right)\left\langle g_{n_{k_1}} - \bar{b}\left(h_{n_{k_1}}-\psi\right),\xi\right\rangle_{L^2(J)} = \lim_{k_2\to\infty}\left(1+\mu_{n_{k_2}}\right)\left\langle g_{n_{k_2}} - \bar{b}\left(h_{n_{k_2}}-\psi\right),\xi\right\rangle_{L^2(J)}$$

$$\Rightarrow \quad \left(\mu^{(1)}-\mu^{(2)}\right)\left\langle \gamma - \bar{b}\,(h-\psi)\,,\xi\right\rangle_{L^2(J)} = 0.$$

As before, because of $h \notin H^2\big|_J$, we derive a contradiction $\mu^{(1)} = \mu^{(2)}$.

Now that the limit in (3.19) exists, we have $\mu \geq -1$. To show $\mu > -1$, assume, by contradiction, that $\mu = -1$. Since $g_n \in \mathcal{B}_{M,h_n}$, for any $\xi \in H^2$, the Cauchy-Schwarz inequality gives

$$\mathrm{Re}\,\langle \psi + b g_n - h_n, \xi\rangle_{L^2(J)} \geq -M\,\|\xi\|_{L^2(J)}\,,$$

and hence it follows from (3.20) (taking real part and passing to the limit as $n \to \infty$) that

$$\underbrace{-(1+\mu)\, M \, \|\xi\|_{L^2(J)}}_{=0} \leq \mathrm{Re}\, \langle f - \psi - b\gamma, \xi \rangle_{L^2(I)}\,,$$

which results in a contradiction since the right-hand side may be made negative due to the assumption that $f \notin \mathcal{A}^{\psi,b}\big|_I$ and to the arbitrary choice of ξ, whereas the left-hand side vanishes by the assumption $\mu = -1$. This finishes the proof of (3.19). ☐

Claim 2. $\gamma \in \mathcal{B}^{\psi,b}_{M,h}$.

Proof. For $g_n \in \mathcal{B}^{\psi,b}_{M,h_n}$, we have $\|\psi + bg_n - h_n\|_{L^2(J)} \leq M$. But $h_n \to h$ in $L^2(J)$, $g_n \rightharpoonup \gamma$ in $L^2(J)$ (as discussed in the proof of Claim 1) and so also $\psi + bg_n - h_n \rightharpoonup \psi + b\gamma - h$ in $L^2(J)$ as $n \to \infty$. The claim now is a direct consequence of the general property of weak limits:

$$\|\tilde{g}\| \leq \liminf_{n\to\infty} \|\tilde{g}_n\| \quad \text{whenever} \quad \tilde{g}_n \rightharpoonup \tilde{g} \quad \text{as} \quad n \to \infty, \tag{3.21}$$

which follows from taking $\xi = \tilde{g}$ in $\lim\limits_{n\to\infty} \langle \tilde{g}_n, \xi\rangle = \langle \tilde{g}, \xi \rangle$ and the Cauchy-Schwarz inequality.

☐

Claim 3. γ is a minimizer of (3.5).

Proof. Since $\gamma \in \mathcal{B}^{\psi,b}_{M,h}$ and g_0 is a minimizer of (3.5), we have

$$\|\psi + bg_0 - f\|_{L^2(I)} \leq \|\psi + b\gamma - f\|_{L^2(I)}\,.$$

To deduce the equality, by contradiction, we assume the strict inequality, or equivalently

$$\|\psi + bg_0 - f\|_{L^2(I)} \leq \|\psi + b\gamma - f\|_{L^2(I)} - \xi \tag{3.22}$$

for some $\xi > 0$. We want to show that this inequality would lead to a contradiction between optimality of solutions $g_0 \in \mathcal{B}^{\psi,b}_{M,h}$ and $g_n \in \mathcal{B}^{\psi,b}_{M,h_n}$ for sufficiently large n.
First of all, there exists $g_0^\star \in \mathcal{B}^{\psi,b}_{M,h}$ and $\tau > 0$ such that

$$\|\psi + bg_0 - f\|_{L^2(I)} = \|\psi + bg_0^\star - f\|_{L^2(I)} - \tau \tag{3.23}$$

and $\|\psi + bg_0^\star - h\|_{L^2(J)} < M$. Indeed, take $g_0^\star = g_0 + \epsilon\delta_g$ with $\delta_g \in H^2$, $\|\delta_g\|_{H^2} = 1$ such that

$$\mathrm{Re}\, \langle \psi + bg_0 - h, b\delta_g \rangle_{L^2(J)} < 0. \tag{3.24}$$

Then, since $\|\psi + bg_0 - h\|_{L^2(J)} = M$ (according to Lemma 1), we have

$$\|\psi + bg_0^\star - h\|^2_{L^2(J)} = \|\psi + bg_0 - h\|^2_{L^2(J)} + 2\epsilon \mathrm{Re}\, \langle \psi + bg_0 - h, b\delta_g\rangle_{L^2(J)} + \epsilon^2 \|\delta_g\|^2_{L^2(J)} = M^2 - \eta_0$$

with $\eta_0 := -2\epsilon \mathrm{Re}\, \langle \psi + bg_0 - h, b\delta_g \rangle_{L^2(J)} - \epsilon^2 \|\delta_g\|^2_{L^2(J)} > 0$ for sufficiently small $\epsilon > 0$, that is

$$\|\psi + bg_0^\star - h\|_{L^2(J)} = M - \eta, \quad \eta := \frac{\eta_0}{\|\psi + bg_0^\star - h\|_{L^2(J)} + M} > 0. \tag{3.25}$$

Now we consider

$$\|\psi + bg_0^\star - f\|^2_{L^2(I)} \quad = \quad \|\psi + bg_0 - f\|^2_{L^2(I)} + 2\epsilon \mathrm{Re}\, \langle \psi + bg_0 - f, b\delta_g\rangle_{L^2(I)} + \epsilon^2 \|\delta_g\|^2_{L^2(I)}$$

and note that the optimality condition (3.10) implies

$$\left\langle \bar{b}\left(f-\psi\right)-g_0,\delta_g\right\rangle_{L^2(I)}=\left(1+\mu\right)\left\langle g_0-\bar{b}\left(h-\psi\right),\delta_g\right\rangle_{L^2(J)}$$

$$\Rightarrow \quad \mathrm{Re}\left\langle \psi+bg_0-f,b\delta_g\right\rangle_{L^2(I)}=-\left(1+\mu\right)\mathrm{Re}\left\langle \psi+bg_0-h,b\delta_g\right\rangle_{L^2(J)}>0,$$

where $\mu>-1$ is the Lagrange parameter for the solution g_0. Therefore,

$$\left\|\psi+bg_0^\star-f\right\|_{L^2(I)}^2=\left\|\psi+bg_0-f\right\|_{L^2(I)}^2+\tau_0$$

with $\tau_0:=-2\left(1+\mu\right)\mathrm{Re}\left\langle \psi+bg_0-h,b\delta_g\right\rangle_{L^2(J)}+\epsilon^2\left\|\delta_g\right\|_{L^2(I)}^2>0$ for small enough ϵ, and so (3.23) follows with

$$\tau:=\frac{\tau_0}{\left\|\psi+bg_0^\star-f\right\|_{L^2(I)}+\left\|\psi+bg_0-f\right\|_{L^2(I)}}>0. \tag{3.26}$$

Now it is easy to see that for large enough n, we also have $g_0^\star\in\mathcal{B}_{M,h_n}^{\psi,b}$. Since $h_n\to h$ in $L^2\left(J\right)$ as $n\to\infty$, there exists $N_1\in\mathbb{N}_+$ such that $\left\|h-h_n\right\|_{L^2(J)}<\eta$ whenever $n>N_1$, so from (3.25), we deduce the bound

$$\left\|\psi+bg_0^\star-h_n\right\|_{L^2(J)}\leq\left\|\psi+bg_0^\star-h\right\|_{L^2(J)}+\left\|h-h_n\right\|_{L^2(J)}\leq M. \tag{3.27}$$

On the other hand, by the property of weak limits (3.21), we have

$$\liminf_{n\to\infty}\left\|\psi+bg_n-f\right\|_{L^2(I)}\geq\left\|\psi+b\gamma-f\right\|_{L^2(I)},$$

that is, for any given $\rho>0$,

$$\left\|\psi+bg_n-f\right\|_{L^2(I)}>\left\|\psi+b\gamma-f\right\|_{L^2(I)}-\rho \tag{3.28}$$

holds when n is taken large enough. In particular, there is $N_2\in\mathbb{N}_+$ such that (3.28) holds for $n\geq N_2$ with $\rho=\tau$. Then, for any $n\geq\max\left\{N_1,N_2\right\}$, (3.28) can be combined with (3.22) and (3.23) to give

$$\left\|\psi+bg_n-f\right\|_{L^2(I)}>\left\|\psi+bg_0^\star-f\right\|_{L^2(I)}+\xi-2\tau.$$

According to (3.26), τ can be made arbitrarily small by the choice of δ_g and ϵ whereas ξ is a fixed number. Therefore, we have $\left\|\psi+bg_0^\star-f\right\|_{L^2(I)}<\left\|\psi+bg_n-f\right\|_{L^2(I)}$ and $g_0^\star\in\mathcal{B}_{M,h_n}^{\psi,b}$ (according to (3.27)). In other words, $g_0^\star$ gives a better solution than g_n, and hence, by uniqueness (Theorem 1), we get a contradiction to the minimality of g_n in (3.16). □

Remark 6. As it is mentioned in the formulation of Theorem 2, for g_0 to be a solution to (3.5), the Lagrange parameter μ has yet to be chosen such that g_0 given by (3.11) satisfies the constraint $\left\|\psi+bg_0-h\right\|_{L^2(J)}=M$, which makes the well-posedness (regularization) effective, see Proposition 4 and discussion in the beginning of Section 5.
We note that the formal substitution $\mu=-1$ in (3.15) leaves out the constraint on J and leads to the situation $\left.g\right|_I=\bar{b}\left(f-\psi\right)$ that was ruled out initially by the requirement (3.7).

When $f\in\left.\mathcal{A}^{\psi,b}\right|_I$, we face an extrapolation problem of holomorphic extension from I inside the disk preserving interior pointwise data. In such a case, $\bar{b}\left(f-\psi\right)\in\left.H^2\right|_I$ and Proposition 1 (or alternative scheme from [31] mentioned in Remark 3) applies to construct the extension g_0 such that $\left.g_0\right|_I=\bar{b}\left(f-\psi\right)$ which can be regarded as the solution if we give up the control on J which means that for a given h the parameter M should be relaxed (yet remaining finite) to avoid an overdetermined problem. Otherwise, keeping the original bound M, despite $f \in \mathcal{A}^{\psi,b}\big|_I$, we must accept non-zero minimum of the cost functional of the problem in which case the solution g_0 is still given by (3.11) which proof is valid since now $g_0|_I \neq \bar{b}\,(f-\psi)$. The latter situation, from geometrical point of view, is nothing but finding a projection of $f \in \mathcal{A}^{\psi,b}\big|_I$ onto the convex subset $\mathcal{C}^{\psi,b}_{M,h} \subseteq \mathcal{A}^{\psi,b}\big|_I$.

However, returning back to the realistic case, when $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}\big|_I$, the solution to (3.5) can still be written in an integral form in spirit of the Carleman's formula (2.11) as given by the following result (see also [6] where it was stated for the case $\psi \equiv 0$, $b \equiv 1$).

Proposition 5. *For $\mu \in (-1,0)$, the solution (3.11) can be represented as*

$$g_0(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \left(\frac{\Phi(\xi)}{\Phi(z)}\right)^{\alpha} \left(\bar{b}\,(f-\psi) \vee \bar{b}\,(h-\psi)\right)(\xi)\,\frac{d\xi}{\xi - z}, \quad z \in \mathbb{D}, \tag{3.29}$$

where

$$\Phi(z) = \exp\left\{\frac{\log\rho}{2\pi i}\int_I \frac{\xi+z}{\xi-z}d\xi\right\}, \quad \alpha = -\frac{\log(1+\mu)}{2\log\rho}, \quad \rho > 1. \tag{3.30}$$

Proof. First of all, we note that (3.30) is a quenching function satisfying $|\Phi| = \rho \vee 1$ on $\mathbb{T}$ and $|\Phi| > 1$ on $\mathbb{D}$ which can be constructed following the recipe of Remark 2. The condition $|\Phi| > 1$ on $\mathbb{D}$ and the minimum modulus principle for analytic functions imply the requirement $\rho > 1$. To show the equivalence, one can start from (3.29) and arrive at (3.11) for a suitable choice of the parameters. Indeed, since $\Phi \in H^\infty$, (3.29) implies

$$\Phi^\alpha g_0 = P_+\left[\Phi^\alpha\left(\bar{b}\,(f-\psi) \vee \bar{b}\,(h-\psi)\right)\right]$$
$$\Rightarrow \quad P_+\left(|\Phi|^{2\alpha} g_0\right) = P_+\left(\bar{\Phi}^\alpha P_+\left[\Phi^\alpha\left(\bar{b}\,(f-\psi) \vee \bar{b}\,(h-\psi)\right)\right]\right).$$

We can represent

$$P_+\left[\Phi^\alpha\left(\bar{b}\,(f-\psi) \vee \bar{b}\,(h-\psi)\right)\right] = \Phi^\alpha\left(\bar{b}\,(f-\psi) \vee \bar{b}\,(h-\psi)\right) - P_-\left[\Phi^\alpha\left(\bar{b}\,(f-\psi) \vee \bar{b}\,(h-\psi)\right)\right]$$

with P_- being anti-analytic projection defined in Section 2. Since

$$\left\langle \bar{\Phi}^\alpha P_-\left[\Phi^\alpha\left(\bar{b}\,(f-\psi) \vee \bar{b}\,(h-\psi)\right)\right], u\right\rangle_{L^2(\mathbb{T})} = \left\langle P_-\left[\Phi^\alpha\left(\bar{b}\,(f-\psi) \vee \bar{b}\,(h-\psi)\right)\right], \Phi^\alpha u\right\rangle_{L^2(\mathbb{T})} = 0$$

for any $u \in H^2$, it follows that $P_+\left(\bar{\Phi}^\alpha P_-\left[\Phi^\alpha\left(\bar{b}\,(f-\psi) \vee \bar{b}\,(h-\psi)\right)\right]\right) = 0$ and so we deduce

$$P_+\left(|\Phi|^{2\alpha} g_0\right) = P_+\left[|\Phi|^{2\alpha}\left(\bar{b}\,(f-\psi) \vee \bar{b}\,(h-\psi)\right)\right].$$

Given $\rho > 1$, choose $\alpha > 0$ such that $\rho^{2\alpha} = \dfrac{1}{1+\mu}$ (this restricts the range $\mu > -1$ to $\mu \in (-1,0)$). Then, $|\Phi|^{2\alpha}\Big|_I = \dfrac{1}{1+\mu}$, $|\Phi|^{2\alpha}\Big|_J = 1$, and hence

$$P_+\left(\frac{1}{1+\mu}g_0 \vee g_0\right) = P_+\left(\frac{\bar{b}}{1+\mu}(f-\psi) \vee \bar{b}\,(h-\psi)\right)$$
$$\Rightarrow \quad P_+(g_0 \vee g_0) + \mu P_+(0 \vee g_0) = P_+\left(\bar{b}\,(f-\psi) \vee (1+\mu)\,\bar{b}\,(h-\psi)\right),$$

which directly furnishes (3.11). $\square$

4 Choice of interpolation function and solution reduction

Before we proceed with computational aspects, it is worth discussing the choice of interpolant ψ which up to this point was any H^2 function satisfying the interpolation conditions (3.1).
We will first consider a particular choice of the interpolant following [35] and then discuss the general case.

Proposition 6. *The H^2 function defined for $z \in \mathbb{D}$ by*

$$\psi(z) = \sum_{k=1}^{N} \psi_k \mathcal{K}(z_k, z) \quad \text{with} \quad \mathcal{K}(z_k, z) := \frac{1}{1 - \bar{z}_k z} \tag{4.1}$$

interpolates the data (3.1) *for an appropriate choice of the constants $\{\psi_k\}_{k=1}^N \in \mathbb{C}$ which exists regardless of a priori prescribed values $\{\omega_k\}_{k=1}^N$ and choice of the points $\{z_k\}_{k=1}^N$ (providing they are all different). Moreover, it is the unique interpolant of minimal H^2 norm.*

Proof. We note that the function $\mathcal{K}(\cdot, \cdot)$ is the reproducing kernel for H^2 meaning that, for any $u \in H^2$, $z_0 \in \mathbb{D}$, point evaluation is given by the inner product $u(z_0) = \langle u, \mathcal{K}(z_0, \cdot)\rangle_{L^2(\mathbb{T})}$, which is a direct consequence of the Cauchy integral formula because $d\theta = \frac{1}{iz}dz$ in (2.7). The coefficients $\{\psi_k\}_{k=1}^N \in \mathbb{C}$ in (4.1) are to be found from the requirement (3.1). We therefore have

$$\psi_k = \sum_{j=1}^{N} S_{kj}\omega_j, \quad \text{where} \quad S := [S_{kj}] = [\mathcal{K}(z_k, z_j)]^{-1}, \; k, j = 1, \dots, N. \tag{4.2}$$

In order to see that the existence of the inverse matrix S is unconditional, we note that $\mathcal{K}(z_k, z_j) = \langle \mathcal{K}(z_k, \cdot), \mathcal{K}(z_j, \cdot)\rangle_{L^2(\mathbb{T})}$, and hence it is the inverse of a Gram matrix which exists since $z_k \neq z_j$ whenever $k \neq j$ providing that all functions $\{\mathcal{K}(z_k, z)\}_{k=1}^N$ are linearly independent. To check the latter, we verify the implication

$$\sum_{k=1}^{N} c_k \mathcal{K}(z_k, z) = 0 \qquad \Rightarrow \qquad c_k = 0, \quad k = 1, \dots, N.$$

Employing the identity $\frac{1}{1 - \bar{z}_k z} = \sum_{n=0}^{\infty} \bar{z}_k^n z^n$ that holds due to $|z_k z| < 1$, we see that

$$\sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k \bar{z}_k^n \right) z^n = 0, \quad \forall z \in \mathbb{D} \qquad \Rightarrow \qquad \sum_{k=1}^{N} c_k \bar{z}_k^n = 0, \quad n \in \mathbb{N}_0.$$

But, by induction on k, this necessarily implies that $c_k = 0$, $k = 1, \dots, N$ and thus proves the linear independence.
To show that $\psi \in H^2$ is the unique interpolant of minimal norm, we let $\psi_0 \in H^2$ be another interpolant satisfying (3.1). Then, $\phi_0 := \psi - \psi_0 \in H^2$ is such that $\phi_0|_{z=z_k} = 0$, $k = 1, \dots, N$, or equivalently,

$$\langle \phi_0, \mathcal{K}(z_k, \cdot)\rangle_{L^2(\mathbb{T})} = 0, \quad k = 1, \dots, N$$

meaning orthogonality of $\phi_0(z)$ to a linear span of $\{\mathcal{K}(z_k, z)\}_{k=1}^N$. But ψ exactly belongs to this span, and hence

$$\|\psi_0\|_{H^2}^2 = \|\psi\|_{H^2}^2 + \|\phi_0\|_{H^2}^2 > \|\psi\|_{H^2}^2, \tag{4.3}$$

which shows that ψ is the unique interpolating H^2 function of minimal norm. □

Remark 7. With this choice of ψ, the solution (3.11) takes the form

$$g_0 = (1+\mu\phi)^{-1}\left[P_+\left(\bar{b}\,(f\vee h)\right) + \mu\, P_+\left(0\vee \bar{b}\,(h-\psi)\right)\right]. \tag{4.4}$$

Indeed, since $\langle \mathcal{K}(z_k,z), bu\rangle_{L^2(\mathbb{T})} = 0$, $k=1,\dots,N$ for any $u\in H^2$, we have $P_+\left(\bar{b}\psi\right)=0$ whenever ψ is given by (4.1).

Now it may look tempting to consider other choices of the interpolant to improve the L^2-bounds in (3.3) or (3.5) rather than being itself of minimal $L^2(\mathbb{T})$ norm. However, the choice of the interpolant does not affect the combination $\tilde{g}_0 = \psi + bg_0$, a result that is not surprising at all from physical point of view since ψ is an auxiliary tool which should not affect solution whose dependence must eventually boil down to given data (measurement related quantities) only: $\{z_k\}_{k=1}^N$, $\{\omega_k\}_{k=1}^N$, f and h. More precisely, we have

Lemma 2. *Given arbitrary ψ_1, $\psi_2\in H^2$ satisfying (3.1), we have $\psi_1 + bg_0(\psi_1) = \psi_2 + bg_0(\psi_2)$.*

Proof. First of all, we note that the dependence $g_0(\psi)$ is not only due to explicit appearance of ψ in (3.11), but also because the Lagrange parameter μ, in general, has to be readjusted according to ψ, that is $\mu=\mu(\psi)$ so that

$$\|\psi_k + bg_0(\psi_k) - h\|^2_{L^2(J)} = M^2, \quad k=1,2, \tag{4.5}$$

where we mean $g_0(\psi) = g_0(\psi,\mu(\psi))$. Let us denote $\delta_\psi := \psi_2-\psi_1$, $\delta_\mu := \mu(\psi_2)-\mu(\psi_1)$, $\delta_g := g_0(\psi_2)-g_0(\psi_1)$. Taking difference of both equations (4.5), we have

$$\langle \delta_\psi + b\delta_g, \psi_1 + bg_0(\psi_1) - h\rangle_{L^2(J)} + \langle \psi_2 + bg_0(\psi_2) - h, \delta_\psi + b\delta_g\rangle_{L^2(J)} = 0$$

$$\Rightarrow \quad 2\mathrm{Re}\left\langle \bar{b}\delta_\psi + \delta_g, \bar{b}\psi_2 + g_0(\psi_2) - \bar{b}h\right\rangle_{L^2(J)} = \|\delta_\psi + b\delta_g\|^2_{L^2(J)}. \tag{4.6}$$

On the other hand, the optimality condition (3.8) implies that, for any $\xi\in H^2$,

$$\left\langle \bar{b}\psi_k + g_0(\psi_k) - \bar{b}f, \xi\right\rangle_{L^2(I)} = -\left(1+\mu(\psi_k)\right)\left\langle \bar{b}\psi_k + g_0(\psi_k) - \bar{b}h, \xi\right\rangle_{L^2(J)}, \quad k=1,2,$$

and therefore

$$\left\langle \bar{b}\delta_\psi + \delta_g, \xi\right\rangle_{L^2(I)} = -\left(1+\mu(\psi_1)\right)\left\langle \bar{b}\delta_\psi + \delta_g, \xi\right\rangle_{L^2(J)} - \delta_\mu\left\langle \bar{b}\psi_2 + g_0(\psi_2) - \bar{b}h, \xi\right\rangle_{L^2(J)}. \tag{4.7}$$

Since $\delta_\psi\in H^2$, due to (3.1), it is zero at each z_j, $j=1,\dots,N$, and hence factorizes as $\delta_\psi = b\eta$ for some $\eta\in H^2$. This allows us to take $\xi = \bar{b}\delta_\psi + \delta_g \in H^2$ in (4.7) to yield

$$\|\eta+\delta_g\|^2_{L^2(I)} = -\left(1+\mu(\psi_1)\right)\|\eta+\delta_g\|^2_{L^2(J)} - \delta_\mu\left\langle \bar{b}\psi_2 + g_0(\psi_2) - \bar{b}h, \eta+\delta_g\right\rangle_{L^2(J)}.$$

Note that the inner product term here is real-valued since the others are, and so employing (4.6), we arrive at

$$\|\eta+\delta_g\|^2_{L^2(I)} + \left(1+\mu(\psi_1)\right)\|\eta+\delta_g\|^2_{L^2(J)} = -\frac{1}{2}\delta_\mu\,\|\eta+\delta_g\|^2_{L^2(J)}$$

which, due to $\mu>-1$, entails that $\delta_\mu\le 0$. But, clearly, interchanging ψ_1 and ψ_2, we would get $\delta_\mu\ge 0$, and so $\delta_\mu = 0$ leading to $\|\delta_\psi + b\delta_g\|^2_{L^2(\mathbb{T})} = \|\eta+\delta_g\|^2_{L^2(I)} + \|\eta+\delta_g\|^2_{L^2(J)} = 0$ which finishes the proof. $\square$

Combining this lemma with Remark 7, we can formulate

Corollary 3. *Independently of choice of $\psi \in H^2$ fulfilling (3.1), the final solution $\tilde{g}_0 = \psi + bg_0$ is given by*

$$\tilde{g}_0 = \psi + b\,(1+\mu\phi)^{-1}\left[P_+\left(\bar{b}\,(f \vee h)\right) + \mu P_+\left(0 \vee \bar{b}\,(h-\psi)\right)\right]. \tag{4.8}$$

These results will be employed for analytical purposes in Section 7.
Even though it is not going to be used here, we also note that it is possible to construct an interpolant whose norm does not exceed *a priori* given bound providing a certain quadratic form involving interpolation data and value of the bound is positive semidefinite [16].

5 Computational issues and error estimate

We would like to stress again that the obtained formulas (3.11), (3.29) and (4.4) furnish solution only in an implicit form with the Lagrange parameter μ still to be chosen such that the solution satisfies the equality constraint in (3.3). As it was mentioned in Remark 6, the constraint in $\mathcal{B}^{\psi,b}_{M,h}$ does not enter the solution characterisation (3.15) when $\mu = -1$, so as $\mu \searrow -1$ we expect perfect approximation of the given $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}\big|_I$ at the expense of uncontrolled growth of the quantity

$$M_0(\mu) := \|\psi + bg_0(\mu) - h\|_{L^2(J)} \tag{5.1}$$

according to Propositions 3 and 4. This is not surprising since the inclusion $\mathcal{B}^{\psi,b}_{M_1,h} \subset \mathcal{B}^{\psi,b}_{M_2,h}$ whenever $M_1 < M_2$ implies that the minimum of the cost functional of (3.5) sought over $\mathcal{B}^{\psi,b}_{M_1,h}$ is bigger than that for $\mathcal{B}^{\psi,b}_{M_2,h}$. For devising a feasible for applications solution, a suitable trade-off between value of μ (governing quality of approximation on I) and choice of the admissible bound M has to be found. To gain insight into this situation, we define the error of approximation as

$$e(\mu) := \|\psi + bg_0(\mu) - f\|^2_{L^2(I)}, \tag{5.2}$$

and proceed with establishing connection between e and M_0.

5.1 Monotonicity and boundedness

Here we mainly follow the steps of [5, 6] where similar studies has been done without interpolation conditions.

Proposition 7. *The following monotonicity results hold*

$$\frac{de}{d\mu} > 0, \qquad \frac{dM_0^2}{d\mu} < 0. \tag{5.3}$$

Moreover, we have

$$\frac{de}{d\mu} = -(\mu+1)\frac{dM_0^2}{d\mu}. \tag{5.4}$$

Proof. From (3.11), using commutation of ϕ and $(1+\mu\phi)^{-1}$, we compute

$$\frac{dg_0}{d\mu} = -(1+\mu\phi)^{-2}\phi P_+\left(\bar{b}(f-\psi) \vee (1+\mu)\bar{b}(h-\psi)\right) + (1+\mu\phi)^{-1} P_+\left(0 \vee \bar{b}(h-\psi)\right)$$

$$\Rightarrow \quad \frac{dg_0}{d\mu} = -(1+\mu\phi)^{-1}\left[\phi g_0 + P_+\left(0 \vee \bar{b}(\psi-h)\right)\right], \tag{5.5}$$

and thus

$$\frac{dM_0^2}{d\mu} = 2\mathrm{Re}\left\langle b\frac{dg_0}{d\mu}, \psi + bg_0 - h\right\rangle_{L^2(J)}$$

$$= -2\mathrm{Re}\left\langle (1+\mu\phi)^{-1}\left[\phi g_0 + P_+\left(0 \vee \bar{b}(\psi-h)\right)\right], \phi g_0 + P_+\left(0 \vee \bar{b}(\psi-h)\right)\right\rangle_{L^2(\mathbb{T})} < 0, \tag{5.6}$$

The inequality here is due to the spectral result (3.13) implying

$$\mathrm{Re}\left\langle (1+\mu\phi)^{-1}\xi, \xi\right\rangle_{L^2(\mathbb{T})} = \left\langle (1+\mu\phi)^{-1}\xi, \xi\right\rangle_{L^2(\mathbb{T})} \geq 0$$

for any $\xi \in H^2$ and $\mu > -1$ whereas the equality in (5.6) would be possible, according to Proposition 2, only when $g_0|_J = \bar{b}(h-\psi)$, that is $M_0 = 0$, the case that was eliminated by Corollary 2.
Now, for any $\beta \in \mathbb{R}$, making use of (5.5) again, we compute

$$\begin{aligned}\frac{de}{d\mu} &= 2\mathrm{Re}\left\langle \frac{dg_0}{d\mu}, \bar{b}\left(\psi-f\right)+g_0\right\rangle_{L^2(I)}\\ &= -2\mathrm{Re}\left\langle \left(1+\mu\phi\right)^{-1}\left[\phi g_0 + P_+\left(0\vee \bar{b}\left(\psi-h\right)\right)\right], \left(\bar{b}\left(\psi-f\right)+g_0\right)\vee 0\right\rangle_{L^2(\mathbb{T})}\\ &= -\beta\frac{dM_0^2}{d\mu} - 2\mathrm{Re}B,\end{aligned}$$

with B given by

$$\begin{aligned}&\left\langle \left(1+\mu\phi\right)^{-1}\left[\phi g_0 + P_+\left(0\vee \bar{b}\left(\psi-h\right)\right)\right], \beta\phi g_0 + \beta P_+\left[0\vee \bar{b}\left(\psi-h\right)\right] + \left(\bar{b}\left(\psi-f\right)+g_0\right)\vee 0\right\rangle_{L^2(\mathbb{T})}\\ &= \left\langle \left(1+\mu\phi\right)^{-1}\left[\phi g_0 + P_+\left(0\vee \bar{b}\left(\psi-h\right)\right)\right], \left(\bar{b}\left(\psi-f\right)+g_0\right)\vee \beta\left[\bar{b}\left(\psi-h\right)+g_0\right]\right\rangle_{L^2(\mathbb{T})}\\ &= \left\langle b\left(1+\mu\phi\right)^{-1}\left[\phi g_0 + P_+\left(0\vee \bar{b}\left(\psi-h\right)\right)\right], \left(\psi+bg_0-f\right)\vee \beta\left(\psi+bg_0-h\right)\right\rangle_{L^2(\mathbb{T})},\end{aligned}$$

where we suppressed the P_+ operator on the right part of the inner product in the second line due to the fact that the left part of it belongs to H^2.
The choice $\beta = \mu+1 = \lambda$ entails $\mathrm{Re}B = 0$ due to (3.8), and we thus obtain (5.4). Since $\mu+1>0$, (5.4) combines with (5.6) to furnish the remaining inequality in (5.3). □

In particular, equation (5.4) encodes how the decay of the approximation error on I is accompanied by $\tilde{g}_0 = \psi + bg_0$ departing further away from given h on J as $\mu \searrow -1$. Even though more concrete asymptotic estimates on the increase of $M_0(\mu)$ near $\mu=-1$ will be discussed later on, we start providing merely a rough square-integrability result which is contained in the following

Proposition 8. *The deviation M_0 of the solution $\tilde{g}_0$ from h on J has moderate growth as $\mu \searrow -1$ so that, for any $-1 < \mu_0 < \infty$,*

$$\int_{-1}^{\mu_0} M_0^2\left(\mu\right)d\mu < \infty. \tag{5.7}$$

Proof. Integration of (5.4) by parts from μ to μ_0 yields

$$e\left(\mu_0\right) - e\left(\mu\right) = \left(\mu+1\right)M_0^2\left(\mu\right) - \left(\mu_0+1\right)M_0^2\left(\mu_0\right) + \int_{\mu}^{\mu_0} M_0^2\left(\tau\right)d\tau. \tag{5.8}$$

As it was already mentioned in the beginning of the section, Proposition 3 implies that the cost functional goes to 0 when μ decays to -1:

$$e\left(\mu\right)\searrow 0 \quad \text{as} \quad \mu\searrow -1. \tag{5.9}$$

We are now going to estimate the behavior of the product $\left(\mu+1\right)M_0^2\left(\mu\right)$. First of all, since the constraint is saturated (Lemma 1), condition (3.10) implies that

$$\begin{aligned}\left\langle f-\psi-bg_0, bg_0\right\rangle_{L^2(I)} &= \left(1+\mu\right)\left\langle h-\psi-bg_0, -bg_0\right\rangle_{L^2(J)}\\ &= \left(1+\mu\right)M_0^2 - \left(1+\mu\right)\left\langle h-\psi-bg_0, h-\psi\right\rangle_{L^2(J)},\end{aligned} \tag{5.10}$$

and therefore

$$e^{1/2}(\mu)\|g_0\|_{L^2(I)} \geq \left|\langle f-\psi-bg_0, bg_0\rangle_{L^2(I)}\right| \geq (1+\mu)\, M_0\left(M_0-\|h-\psi\|_{L^2(J)}\right).$$

Now, since $M_0 \nearrow \infty$ as $\mu \searrow -1$ (because of (5.9) and Proposition 3), the first term is dominant, and thus the right-hand side remains positive. Then, because of (5.9) and finiteness of $\|g_0\|_{L^2(I)}$ (by the triangle inequality, $\|g_0\|_{L^2(I)} \leq e^{1/2}(\mu)+\|\psi-f\|_{L^2(I)}$), we conclude that

$$(\mu+1)\, M_0^2 \searrow 0 \quad \text{as} \quad \mu \searrow -1, \tag{5.11}$$

which allows us to deduce (5.7) from (5.8). □

Remark 8. In the simplified case with no pointwise interpolation conditions (or those of zero-values) and no information on J, the conclusion of the Proposition can be strengthened to

$$\|M_0\|_{L^2(-1,\infty)} := \left(\int_{-1}^{\infty} M_0^2(\mu)\, d\mu\right)^{1/2} = \|f\|_{L^2(I)}, \tag{5.12}$$

a result that was given in [5]. This mainly relies on the fact that, for $\psi \equiv 0$ and $h \equiv 0$,

$$g_0 \to 0 \quad \text{in} \quad L^2(\mathbb{T}) \quad \text{as} \quad \mu \nearrow \infty, \tag{5.13}$$

which holds by the following argument. Denoting $\tilde{f} := P_+\left(\bar{b}f \vee 0\right)$, the solution formulas (3.11) and (3.15) become $g_0 = (1+\mu\phi)^{-1}\tilde{f}$ and $\mu\phi g_0 = \tilde{f}-g_0$, respectively. From these, as $\mu \nearrow \infty$, using the spectral theorem (see Appendix), we obtain

$$\|\phi g_0\|_{H^2} = \frac{1}{\mu}\left\|\tilde{f}-g_0\right\|_{H^2} \leq \frac{1}{\mu}\|f\|_{L^2(I)}\left[1+\left\|(1+\mu\phi)^{-1}\right\|\right] \leq \frac{2}{\mu}\|f\|_{L^2(I)} \searrow 0,$$

and hence, by Proposition 2, conclude that $\|g_0\|_{H^2} \searrow 0$. We also need to show that

$$(\mu+1)\, M_0^2 \searrow 0 \quad \text{as} \quad \mu \nearrow \infty, \tag{5.14}$$

but this follows from the positivity $(\mu+1)\, M_0^2 > 0$ and the observation that, for large enough μ, we have

$$\frac{d\left[(\mu+1)\, M_0^2\right]}{d\mu} = M_0^2 + (\mu+1)\frac{dM_0^2}{d\mu} < 0$$

(the inequality holds since, due to (5.6), the second term in the right-hand side is strictly negative whereas the first one goes to zero as μ increases). Finally, further elaboration of (5.10) into

$$e(\mu) + (1+\mu)\, M_0^2(\mu) = \langle \psi+bg_0-f, \psi-f\rangle_{L^2(I)} + (1+\mu)\langle \psi+bg_0-h, \psi-h\rangle_{L^2(J)}$$

yields, in the case $\psi \equiv 0$, $h \equiv 0$,

$$e(\mu) + (1+\mu)\, M_0^2(\mu) = \langle f-bg_0, f\rangle_{L^2(I)},$$

which, by (5.13)-(5.14), furnishes $\lim\limits_{\mu\to\infty} e(\mu) = \|f\|^2_{L^2(I)}$, and hence (5.12) follows from (5.8) recalling again (5.9) and (5.11).

5.2 Sharper estimates

Precise asymptotic estimates near $\mu=-1$ were obtained in [7] using concrete spectral theory of Toeplitz operators [32, 33]. Namely, under some specific regularity assumptions on the boundary data f (related to integrability of the first derivative on I), we have

$$M_0^2(\mu)=\mathcal{O}\left((1+\mu)^{-1}\log^{-2}(1+\mu)\right),\quad e(\mu)=\mathcal{O}\left(\left|\log^{-1}(1+\mu)\right|\right)\qquad \text{as}\quad \mu\searrow -1. \tag{5.15}$$

Here we suggest a way of *a priori* estimation of approximation rate and error bounds without resorting to an iterative solution procedure. This is based on a Neumann-like expansion of the inverse Toeplitz operator which provides series representations for the quantities $e(\mu)$ and $M_0^2(\mu)$ for values of μ moderately greater than -1 and, therefore, complements previously obtained estimates of the asymptotic behavior of these quantities in the vicinity of $\mu=-1$. Moreover, using these series expansions, we further attempt to recover the estimates (5.15) without having concrete spectral theory involved, yet still appealing to some general spectral theory results.
It is convenient to introduce the quantity

$$\xi(\mu):=\phi g_0(\mu)+P_+\left(0\vee\bar{b}(\psi-h)\right) \tag{5.16}$$

that enters equation (5.5). The main results will be obtained in terms of

$$\xi_0:=\xi(0)=\phi\left(P_+\left(\bar{b}(f-\psi)\vee\bar{b}(h-\psi)\right)\right)-P_+\left(0\vee\bar{b}(h-\psi)\right). \tag{5.17}$$

Proposition 9. *For $|\mu|<1$, we have*

$$M_0^2(\mu)=M_0^2(0)-\sum_{k=0}^{\infty}(-1)^k(k+2)F(k)\mu^{k+1}, \tag{5.18}$$

$$e(\mu)=e(0)+2\sum_{k=0}^{\infty}(-1)^kF(k)\mu^{k+1}+\sum_{k=1}^{\infty}(-1)^kk\left[F(k)-F(k-1)\right]\mu^{k+1}, \tag{5.19}$$

where $F(k):=\left\langle\phi^k\xi_0,\xi_0\right\rangle_{L^2(\mathbb{T})}$, $k\in\mathbb{N}_+$.

Proof. Consider, for $k\in\mathbb{N}_+$, $\mu>-1$,

$$A_k(\mu):=\left\langle(1+\mu\phi)^{-k}\phi^{k-1}\xi(\mu),\xi(\mu)\right\rangle_{L^2(\mathbb{T})}.$$

Since $\xi'(\mu)=\phi\dfrac{dg_0}{d\mu}=-(1+\mu\phi)^{-1}\phi\xi(\mu)$ (according to (5.5)), it follows that

$$\begin{aligned}A_k'(\mu) &= -k\left\langle(1+\mu\phi)^{-k-1}\phi^k\xi(\mu),\xi(\mu)\right\rangle_{L^2(\mathbb{T})}-\left\langle(1+\mu\phi)^{-k-1}\phi^k\xi(\mu),\xi(\mu)\right\rangle_{L^2(\mathbb{T})}\\ &\quad-\left\langle(1+\mu\phi)^{-k}\phi^{k-1}\xi(\mu),(1+\mu\phi)^{-1}\phi\xi(\mu)\right\rangle_{L^2(\mathbb{T})},\end{aligned}$$

and we thus arrive at the infinite-dimensional linear dynamical system

$$\begin{cases}A_k'(\mu) &= -(k+2)A_{k+1}(\mu),\\ A_k(0) &= \left\langle\phi^{k-1}\xi_0,\xi_0\right\rangle_{L^2}=:F(k-1),\end{cases}\qquad k\in\mathbb{N}_+. \tag{5.20}$$

Introduce the matrix $\mathcal{M}$ whose powers are upper-diagonal with evident structure

$$\mathcal{M} = \begin{pmatrix} 0 & -3 & 0 & 0 & \dots \\ 0 & 0 & -4 & 0 & \dots \\ 0 & 0 & 0 & -5 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \mathcal{M}^2 = \begin{pmatrix} 0 & 0 & (-3)(-4) & 0 & \dots \\ 0 & 0 & 0 & (-4)(-5) & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad \dots,$$

which makes the matrix exponential $e^{\mathcal{M}}$ easily computable. Then, due to such a structure, the system (5.20) is readily solvable, but of particular interest is the first component of the solution vector

$$A_1(\mu) \quad = \quad \sum_{k=1}^{\infty} \left[e^{\mathcal{M}\mu}\right]_{1,k} F(k-1) = \sum_{k=0}^{\infty} (-1)^k \frac{(k+2)!}{2} \frac{\mu^k}{k!} F(k),$$

where the series converges for $|\mu| < 1$ since $F(k)$ is bounded by $\|\xi_0\|^2_{H^2} = A_1(0) = F(0)$, as the Toeplitz operator ϕ is a contraction: $F(k)$ slowly decays to zero with k (see also plots and discussion at the end of Section 8).

On the other hand, observe that, due to (5.6), $A_1(\mu) = -\dfrac{1}{2}\dfrac{dM_0^2}{d\mu}$ and thus

$$\frac{dM_0^2}{d\mu} = -\sum_{k=0}^{\infty} (-1)^k (k+1)(k+2) \mu^k F(k). \tag{5.21}$$

Finally, termwise integration of (5.21) and use of (5.4) followed by rearrangement of terms furnish the results (5.18)-(5.19). □

Remark 9. Note that when set $\psi \equiv 0$, $h \equiv 0$, it is seen that (5.21) can be obtained directly from (3.11), (5.6) which now reads

$$\frac{dM_0^2}{d\mu} = -2\mathrm{Re}\left\langle (1+\mu\phi)^{-3} \phi^2 P_+\left(\bar{b}f \vee 0\right), P_+\left(\bar{b}f \vee 0\right)\right\rangle_{L^2(\mathbb{T})}.$$

The result follows since a Neumann series (defining an analytic function for $|\mu| < 1$) is differentiable:

$$(1+\mu\phi)^{-1} = \sum_{k=0}^{\infty} (-1)^k \mu^k \phi^k \quad \Rightarrow \quad (1+\mu\phi)^{-3} = \frac{1}{2}\sum_{k=0}^{\infty} (-1)^k (k+1)(k+2) \mu^k \phi^k.$$

We can also get some insight in behavior of $F(k)$ which lies in the heart of the series expansions (5.18)-(5.19) that will allow us to infer the bounds (5.15). First, we need the following

Lemma 3. *The sequence $\{F(k)\}_{k=0}^{\infty}$ is Abel summable*[2] *and it holds true that*

$$\lim_{\mu \to -1} \sum_{k=0}^{\infty} (-\mu)^k F(k) = e(0) < \infty. \tag{5.22}$$

[2] By such summability we mean that $\sum_{k=0}^{\infty} \mu^k F(k)$ converges for all $|\mu| < 1$ and the limit $\lim\limits_{\mu \nearrow 1} \sum_{k=0}^{\infty} \mu^k F(k)$ exists and is finite.

Proof. Set $R_\mu(N) := \sum_{k=1}^N [F(k) - F(k-1)]\, k\, (-\mu)^k$ and apply summation by parts formula

$$R_\mu(N) = F(N)(N+1)(-\mu)^{N+1} + \mu F(0) - \sum_{k=1}^N F(k)\left((-\mu)^{k+1}(k+1) - (-\mu)^k k\right).$$

Passing to the limit and rearranging the terms, we obtain

$$\lim_{N\to\infty} R_\mu(N) = -\sum_{k=0}^\infty (-\mu)^{k+1} F(k) + (\mu+1)\sum_{k=1}^\infty (-\mu)^k kF(k),$$

and hence it follows from (5.19) that

$$e(\mu) = e(0) + (\mu+2)\sum_{k=0}^\infty (-\mu)^{k+1} F(k) + (\mu+1)\sum_{k=1}^\infty (-\mu)^{k+1} kF(k). \tag{5.23}$$

Combining the local integrability of $M_0^2(\mu)$, equivalent to (5.11), with the series expansion (5.18), we conclude that:

$$(\mu+1)\sum_{k=1}^\infty (-\mu)^k kF(k) \to 0 \text{ as } \mu \searrow -1.$$

Therefore, taking the limit $\mu \searrow -1$ in (5.23), the result (5.22) follows due to (5.9). □

Now, without getting into detail of concrete spectral theory of Toeplitz operators, we only employ existence of a unitary transformation $U: H^2 \to L^2_\lambda(\sigma)$ onto the spectral space where the Toeplitz operator is diagonal, meaning that its action simply becomes a multiplication by the spectral variable λ. Existence of such an isometry along with information on the spectrum of ϕ (Hartman-Wintner theorem, see Appendix), $\sigma = [0,1]$, and an assumption on the constant spectral density[3] $\rho_0 > 0$ make the following representation possible

$$F(k) = \int_0^1 \lambda^k \left|(U\xi_0)(\lambda)\right|^2 \rho_0 d\lambda \tag{5.24}$$

with $\int_0^1 |(U\xi_0)(\lambda)|^2 \rho_0 d\lambda = \|\xi_0\|^2_{H^2}$.
All the essential information on asymptotics (5.15) is contained in behavior of $(U\xi_0)(\lambda)$ near $\lambda = 1$. Even though $(U\xi_0)(\lambda)$ can be computed since ξ_0 is a fixed function defined by (5.17) and the concrete spectral theory describes explicit action of the transformation U [7, 33], we avoid these details and proceed by deriving essential estimates invoking only rather intuitive arguments on the behavior of the resulting function $(U\xi_0)(\lambda)$.
Considering $-1 < \mu < 0$ in what follows, we, first of all, claim that the function $(U\xi_0)(\lambda)$ must necessarily decrease to zero as $\lambda \nearrow 1$. Indeed, even though L^2-behavior allows to have an integrable singularity at $\lambda = 1$, we note that even if regularity was assumed, that is $\lim_{\lambda\to 1} |(U\xi_0)(\lambda)|^2 = C$ for some $C > 0$, after summation of a geometric series, we would have

$$\frac{1}{\rho_0}\sum_{k=0}^\infty (-\mu)^k F(k) \geq C_0 \sum_{k=0}^\infty \int_{1-\delta}^1 (-\mu\lambda)^k d\lambda = C_0 \int_{1-\delta}^1 \frac{1}{1+\mu\lambda} d\lambda = \frac{C_0}{\mu}\log\left(\frac{1+\mu}{1+\mu-\mu\delta}\right)$$

[3]Such an assumption is reasonable since the operator symbol χ_J is the simplest in a sense that it does not differ from one point to another in the region where it is non-zero and therefore the spectral mapping is anticipated to be uniform. Precise expression for the constant ρ_0 can be found in [7, 32].

for some $0 < C_0 \leq C$ and sufficiently small fixed $\delta > 0$. The right-hand side here grows arbitrary large as μ comes closer to -1 contradicting the boundedness prescribed by Lemma 3. Therefore, the decay to zero of $(U\xi_0)(\lambda)$ as $\lambda \nearrow 1$ is necessary.

Next, it is natural to proceed by checking if a very mild (meaning slower than any power) decay to zero can be reconciled with the previously obtained results. Namely, we consider $(U\xi_0)(\lambda)$ such that

$$|(U\xi_0)(\lambda)|^2 = \mathcal{O}\left(|\log(1-\lambda)|^{-l}\right) \quad \text{as} \quad \lambda \nearrow 1, \tag{5.25}$$

for $l > 1$. This entails the following result generalizing (5.15), see also Remarks 10, 11.

Proposition 10. *Under assumption (5.25) with $l > 1$, the solution blow-up and approximation rates near $\mu = -1$, respectively, are as follows*

$$M_0^2(\mu) = \mathcal{O}\left(\frac{1}{1+\mu}|\log(1+\mu)|^{-l}\right), \quad e(\mu) = \mathcal{O}\left(|\log(1+\mu)|^{-l+1}\right). \tag{5.26}$$

Proof. Choose a constant $0 < \lambda_0 < 1$ sufficiently close to 1 so that the asymptotic (5.25) is applicable. Therefore, we can write

$$\frac{1}{\rho_0}\sum_{k=0}^{\infty}(-\mu)^k F(k) = S_1 + S_2 + S_3$$

$$:= \int_0^{\lambda_0}\frac{1}{1+\mu\lambda}|(U\xi_0)(\lambda)|^2 d\lambda + \left(\int_{\lambda_0}^{1-\delta_0} + \int_{1-\delta_0}^{1}\right)\frac{1}{1+\mu\lambda}(-\log(1-\lambda))^{-l} d\lambda\,.$$

The first integral here is bounded regardless of the value of μ:

$$S_1 \leq \frac{1}{1+\mu\lambda_0}\int_0^1 |(U\xi_0)(\lambda)|^2 d\lambda = \frac{1}{(1+\mu\lambda_0)\rho_0}\|\xi_0\|_{H^2}^2\,.$$

To deal with S_3, we perform the change of variable $\beta = -\log(1-\lambda)$ and bound the factor $\frac{1}{\beta^l} \leq (-\log\delta_0)^{-l}$ to obtain

$$\int_{-\log\delta_0}^{\infty}\frac{1}{\beta^l}\frac{e^{-\beta}}{1+\mu-\mu e^{-\beta}}d\beta \leq \frac{1}{(-\mu)(-\log\delta_0)^l}\log\left(1-\frac{\mu\delta_0}{1+\mu}\right) \leq \frac{\log 2}{(-\mu)|\log(1+\mu)-\log(-\mu)|^l}$$

providing we choose $\delta_0 \leq \frac{1+\mu}{(-\mu)}$. The quantity on the right is $\mathcal{O}\left(|\log(1+\mu)|^{-l}\right)$ in the vicinity of $\mu = -1$.

It remains to estimate S_2. The change of variable $\eta = 1-\lambda$ leads to

$$\begin{aligned} S_2 &= \int_{\delta_0}^{1-\lambda_0}\frac{\eta}{1+\mu-\mu\eta}\frac{1}{\eta(-\log\eta)^l}d\eta \leq \left(\int_{\delta_0}^{1-\lambda_0}\frac{d\eta}{\eta(-\log\eta)^l}\right)\sup_{\eta\in[\delta_0,1-\lambda_0]}\left(\frac{\eta}{1+\mu-\mu\eta}\right) \\ &\leq \frac{1}{l-1}\left(\frac{1}{|\log\delta_0|^{l-1}} - \frac{1}{|\log(1-\lambda_0)|^{l-1}}\right)\frac{1-\lambda_0}{1+\mu\lambda_0}. \end{aligned}$$

Therefore, we conclude that the choice (5.25) with $l > 1$ does not contradict the finiteness imposed by Lemma 3 anymore and we move on to obtain the growth rate for $M_0^2(\mu)$ near $\mu = -1$. Recalling (5.18) and that $\sum_{k=0}^{\infty}(-\mu\lambda)^k(k+1) = \frac{1}{(1+\mu\lambda)^2}$, we now have

$$\frac{1}{\rho_0}\sum_{k=0}^{\infty}(-\mu)^k(k+1)F(k) = R_1 + R_2 + R_3 + R_4$$

$$:= \int_0^{\lambda_0} \frac{1}{(1+\mu\lambda)^2} \left|(U\xi_0)(\lambda)\right|^2 d\lambda + \left(\int_{\lambda_0}^{1-\delta_1} + \int_{1-\delta_1}^{1-\delta_2} + \int_{1-\delta_2}^{1} \right) \frac{1}{(1+\mu\lambda)^2} \left(-\log(1-\lambda)\right)^{-l} d\lambda\,.$$

As before, we estimate

$$R_1 \leq \frac{1}{(1+\mu\lambda_0)^2} \int_0^1 \left|(U\xi_0)(\lambda)\right|^2 d\lambda = \frac{1}{(1+\mu\lambda_0)^2 \rho_0} \|\xi_0\|_{H^2}^2\,,$$

whereas the rest is now split into 3 parts and we start with the last term and decide on proper size of δ_2 in

$$R_4 = \int_{1-\delta_2}^{1} \frac{1}{(1+\mu\lambda)^2} \frac{1}{\left|-\log(1-\lambda)\right|^l} d\lambda.$$

Again, under the change of variable $\beta = -\log(1-\lambda)$, this becomes

$$\begin{aligned} R_4 &= \frac{1}{(1+\mu)^2} \int_{-\log\delta_2}^{\infty} \frac{e^{-\beta}}{\beta^l} \frac{1}{\left(1 - \frac{\mu}{1+\mu} e^{-\beta}\right)^2} d\beta \\ &= \frac{1}{(1+\mu)^2} \sum_{k=0}^{\infty} \left(\frac{\mu}{1+\mu}\right)^k (k+1) \int_{-\log\delta_2}^{\infty} \frac{e^{-(k+1)\beta}}{\beta^l} d\beta, \end{aligned}$$

where the series expansion is valid for $\delta_2 < \dfrac{1+\mu}{(-\mu)}$. The integral on the right is the incomplete gamma function (see, for instance, [2]) whose asymptotic expansion for large values of $(-\log\delta_2)$ can be easily obtained with integration by parts. In particular, at the leading order we have

$$\begin{aligned} \int_{-\log\delta_2}^{\infty} \frac{e^{-(k+1)\beta}}{\beta^l} d\beta &= (k+1)^{l-1} \int_{-(k+1)\log\delta_2}^{\infty} \frac{e^{-\beta}}{\beta^l} d\beta \\ &= (k+1)^{l-1} \delta_2^{k+1} \left(-(k+1)\log\delta_2\right)^{-l} \left[1 + \mathcal{O}\left(\frac{1}{(k+1)\left|\log\delta_2\right|}\right)\right], \end{aligned}$$

and hence

$$R_4 = \frac{\delta_2}{(1+\mu)^2} \frac{1}{(-\log\delta_2)^l} \sum_{k=0}^{\infty} \left(\frac{\mu\delta_2}{1+\mu}\right)^k = \frac{\delta_2}{(1+\mu)^2} \frac{1}{(-\log\delta_2)^l} \frac{1}{1 - \frac{\mu\delta_2}{1+\mu}}.$$

Fixing $\delta_2 = \dfrac{1}{2}\dfrac{1+\mu}{(-\mu)}$, we arrive at

$$R_4 = \frac{1}{(-\mu)(1+\mu)} \frac{1}{\left[-\log(1+\mu) + \log(-\mu) + \log 2\right]^l}.$$

To estimate R_2 and R_3, we use change of variable $\eta = 1-\lambda$. Similarly to S_2, we have

$$R_2 = \int_{\delta_1}^{1-\lambda_0} \frac{\eta}{(1+\mu-\mu\eta)^2} \frac{1}{\eta(-\log\eta)^l} d\eta \leq \left(\int_{\delta_1}^{1-\lambda_0} \frac{d\eta}{\eta(-\log\eta)^l} \right) \sup_{\eta\in[\delta_1, 1-\lambda_0]} \left(\frac{\eta}{\left[1+\mu-\mu\eta\right]^2} \right),$$

however, now under the supremum sign, instead of a monotonic function, we have an expression that attains a maximum value $\dfrac{1}{4(-\mu)(1+\mu)}$ if $\delta_1 < \dfrac{1+\mu}{(-\mu)}$ which lacks the smallness we obtained in R_4. Therefore, to remedy the situation, we require $\delta_1 > \dfrac{1+\mu}{(-\mu)}$ and obtain

$$R_2 \leq \frac{1}{l-1}\left(\frac{1}{|\log \delta_1|^{l-1}} - \frac{1}{|\log(1-\lambda_0)|^{l-1}}\right)\frac{\delta_1}{(1+\mu-\mu\delta_1)^2} = \mathcal{O}\left(\frac{1}{1+\mu}\,|\log(1+\mu)|^{-\gamma}\right)$$

near $\mu=-1$, if we fix $\delta_1 = \dfrac{1+\mu}{(-\mu)}\left(1+[-\log(1+\mu)]^{\gamma}\right)$ for arbitrary $\gamma>0$.
The last integral R_3 is to bridge the gap between the two neighborhoods of $\lambda=1$:

$$R_3 = \int_{\delta_2}^{\delta_1} \frac{1}{(1+\mu-\mu\eta)^2}\frac{1}{(-\log\eta)^l}d\eta \leq \frac{1}{(-\log\delta_1)^l}\left(\frac{1}{1+\mu-\mu\delta_2} - \frac{1}{1+\mu-\mu\delta_1}\right)$$

and hence, using the fact that $\log(-\log(1+\mu)) = o(-\log(1+\mu))$, we deduce that near $\mu=-1$

$$R_3 = \mathcal{O}\left(\frac{1}{1+\mu}\,|\log(1+\mu)|^{-l}\right).$$

Now that all the integral terms are estimated, choice of the parameter $\gamma = l$ in δ_1 leads to the first estimate in (5.26) whereas integration of (5.4) recovers the second one. □

Remark 10. The case $l=2$ gives exactly the expressions in (5.15). The assumed behavior (5.25) of $(U\xi_0)(\lambda)$ is analogous (with direct correspondence in the case $\psi\equiv 0$, $h\equiv 0$) to the conclusion of [7, Prop. 4.1] which was used to generate further estimates therein, and the case $l=3$ is related to improved estimates given in [7, Cor. 4.6] under assumption of even higher regularity of boundary data (roughly speaking, integrability of second derivatives). It is noteworthy that the choice $l=1$ yields non-integrable behavior of $M_0^2(\mu)$ contradicting Proposition 8, and therefore was eliminated in the formulation. This is not due to the fact that the method of estimation of the S_2 integral fails, but because of non-integrability near $\mu=-1$ of the overall bound. The R_4 term has been computed asymptotically sharply though it could be made even smaller by shrinking the neighborhood δ_2. Indeed, instead of the $\dfrac{1}{2}$ factor in δ_2, we could have put $\dfrac{1}{1+[-\log(1+\mu)]^{\beta}}$ for any $\beta\geq 0$ similarly to what was done in the R_2 term which allowed a multiplier with arbitrary logarithmical smallness regulated by the parameter γ. This, however, would not reduce the overall blow-up because of the stiff bridging term R_3. Even though the estimate for R_3 is rough, we do not expect improvement by an order of magnitude because the logarithmic factor of the integrand picks up $(1+\mu)$ as a major multiplier near $\eta=\delta_1$ which makes any choice of $\gamma\geq l$ and $\beta\geq 0$ useless in attempt to improve the smallness factor in the blow-up of $M_0^2(\mu)$.

Remark 11. Generally, we note that the appearance of the $\log(1+\mu)$ factors in the bounds is not accident, but intrinsically encoded in the connection between $e(\mu)$ and $M_0^2(\mu)$ since (5.4) can be rewritten as $e'(\mu) = -\dfrac{dM_0^2}{d[\log(1+\mu)]}$ which also explains the choice of (5.25).

We would like to point out again that even though our reasoning was meant to provide an intuitive explanation of the estimates (5.15), more rigourous proofs can be found in [7] where an elegant connection of the bounds with regularity of given boundary data is established by elaborating concrete spectral theory results [33] into formulation of a certain integral transformation followed by application of L^1-theory of Fourier transforms (Riemann-Lebesgue lemma). Also, one can take an alternative viewpoint based on the results of [32]. In that case, the unitary transformation U diagonalizing the Toeplitz operator ϕ acts on Fourier coefficients $\{\eta_n\}_{n=0}^{\infty} \in l^2(\mathbb{N}_0)$ of a given $\xi_0 \in H^2$ as

$$(U\xi_0)(\lambda) = \sum_{n=0}^{\infty} \eta_n \psi_n(\lambda), \tag{5.27}$$

where the orthonormal sequence of $L^2(0,1)$ functions $\psi_n(\lambda)$ are explicitly defined in terms of the Meixner-Pollaczek polynomials of order 1/2 [30]:

$$\psi_n(\lambda) := e^{a\beta}\left(1+e^{-2\pi\beta}\right)^{1/2} P_n^{(1/2)}(\beta, a), \qquad \beta := -\frac{1}{2\pi}\log\left(\frac{1}{\lambda}-1\right)$$

providing $I = \left(e^{-ia}, e^{ia}\right)$, $a \in (0,\pi)$, an assumption that does not reduce the generality if the original sets I and J are two disjoint arcs.
A recurrence formula for the Meixner-Pollaczek polynomials follows from that for the Pollaczek polynomials [36]:

$$P_n^{(1/2)}(\beta) = \frac{1}{n}\left(2\beta\sin a - (2n-1)\cos a\right) P_{n-1}^{(1/2)}(\beta) - \frac{n-1}{n} P_{n-2}^{(1/2)}(\beta), \tag{5.28}$$

$$P_{-1}^{(1/2)}(\beta) = 0, \quad P_0^{(1/2)}(\beta) = 1,$$

which allows to generate all the coefficients $k_m^{(n)}$ in $P_n^{(1/2)}(\beta) = \sum_{m=0}^{n} k_m^{(n)} \beta^m$, for instance,

$$k_n^{(n)} = \frac{(2\sin a)^n}{n!}, \quad k_{n-1}^{(n)} = -n\cos a \frac{(2\sin a)^{n-1}}{(n-1)!},$$

$$k_{n-2}^{(n)} = \frac{1}{6}\left[3n(n-1)\cos^2 a - (2n-1)\sin^2 a\right] \frac{(2\sin a)^{n-2}}{(n-2)!}.$$

Rearranging the terms in (5.27), we can write (suppressing the first two factors for the sake of compactness)

$$\begin{aligned}
(U\xi_0)(\lambda) &\propto \sum_{n=0}^{\infty} \eta_n \sum_{m=0}^{n} k_m^{(n)} \beta^m = \sum_{m=0}^{\infty}\left(\sum_{n=m}^{\infty} \eta_n k_m^{(n)}\right)\beta^m \\
&= \sum_{m=0}^{\infty}\left(\eta_m k_m^{(m)} + \eta_{m+1} k_m^{(m+1)} + \dots\right)\beta^m. \qquad (5.29)
\end{aligned}$$

It would be interesting to see, in such a representation, what decay assumptions on the Fourier coefficients η_n are consistent with (5.25), and thus (5.26), with $1 < l < 2$ in which case there is no violation of integrability of $M_0^2(\mu)$ and less regularity assumptions (namely, milder than decay of $n\eta_n$ to zero as $n \to \infty$) are expected than those related with integrability of the first derivative of boundary data.
Note that, because of the Taylor series of the exponential function, we have

$$\begin{aligned}
\left|\sum_{m=0}^{\infty}\left(\eta_m k_m^{(m)}\right)\beta^m\right| &\leq \left(\sup_{m\in\mathbb{N}_0}|\eta_m|\right)\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\sin a}{\pi}\right)^m \left|\log\left(\frac{1}{\lambda}-1\right)\right|^m \\
&= \left(\sup_{m\in\mathbb{N}_0}|\eta_m|\right)\begin{cases}\left(\frac{1}{\lambda}-1\right)^{\frac{\sin a}{\pi}}, & 0<\lambda<\frac{1}{2}, \\ \left(\frac{1}{\lambda}-1\right)^{-\frac{\sin a}{\pi}}, & \frac{1}{2}\leq\lambda<1,\end{cases}
\end{aligned}$$

and thus the very first term already adds to the singular behavior of (5.27) near $\lambda = 1$ (unless additional assumptions on alternation of sign of η_m are made) instead of revealing any decay to zero. This suggests that terms in the brackets of (5.29) should not be estimated separately: the other terms contribute equally to $(U\xi_0)(\lambda)$ though their expressions are much more cumbersome for straightforward analysis.
An alternative way might be to work in direction of obtaining estimates of (5.18)-(5.19) near $\mu = -1$ in terms of η_m from

$$\int_0^1 \frac{1}{1+\mu\lambda} \left|(U\xi_0)(\lambda)\right|^2 d\lambda \quad \text{and} \quad \int_0^1 \frac{1}{(1+\mu\lambda)^2} \left|(U\xi_0)(\lambda)\right|^2 d\lambda$$

directly without deducing behavior of $(U\xi_0)(\lambda)$ in vicinity of $\lambda = 1$, but using explicit form of the unitary transformation (5.27). To take advantage of it, one can potentially expand integrand factors $\frac{1}{1+\mu\lambda}$ in terms of β and iteratively employ the recurrence formula (5.28) rewritten as

$$\beta P_n^{(1/2)}(\beta, a) = \frac{n+1}{2\sin a} P_{n+1}^{(1/2)}(\beta, a) + \frac{(2n+1)\cot a}{2} P_n^{(1/2)}(\beta, a) + \frac{n}{2\sin a} P_{n-1}^{(1/2)}(\beta, a)$$

followed by application of orthonormality. Note that such a strategy (but based on expansion of λ in terms of β) along with the fact that $U^{-1}\psi_n(\lambda) = z^n$ might also be used to see how the Toeplitz operator ϕ acts on Fourier coefficients of a function.

6 Companion problem

At this moment, it is time to point out a link with another bounded extremal problem which relies on the observation that formal substitution of $\mu = 0$ in (4.8) implies that

$$\tilde{g}_0 = \psi + bP_+\left(\bar{b}\left(f \vee h\right)\right) \tag{6.1}$$

is an explicit solution for the problem with the particular constraint

$$M = M_0\left(0\right) = \left\|\psi + bP_+\left(\bar{b}\left(f \vee h\right)\right) - h\right\|_{L^2(J)}.$$

Recalling that $bP_+\bar{b}$ is a projector onto bH^2 (see Section 2), we note that, geometrically, the solution (6.1) is simply a realization of projection of $f \vee h \in L^2\left(\mathbb{T}\right)$ onto $\mathcal{C}^{\psi,b}_{M,h}$. Now, exploiting the arbitrariness of choice of interpolant ψ (Remark 7), we can change our viewpoint and look for $\psi \in H^2$ meeting pointwise constraints (3.1) such that $\psi + bP_+\left(\bar{b}\left(f \vee h\right)\right) - h$ is sufficiently close to the constant[4] $M/\sqrt{|J|}$ in $L^2\left(J\right)$ yet remaining L^2-bounded on I. In other words, given arbitrary $\psi_0 \in H^2$ satisfying the pointwise interpolation conditions (3.1) (take, for instance, (4.1)), we represent $\psi = \psi_0 + b\Psi$ and thus search for an approximant $\Psi \in H^2$ to $\bar{b}\left(h - \psi_0 + M\right) - P_+\left(\bar{b}\left(f \vee h\right)\right) \in L^2\left(J\right)$ such that $\|\Psi\|_{L^2(I)} = K$ for arbitrary $K \in (0, \infty)$. We thus reduce the original problem to an associated approximation problem on J for which all known data are now prescribed on J alone. Since the constraint on I is especially simple (role of ψ and h play identically zero functions), such a companion problem has a computational advantage over the original one as, due to the form of solution (3.11), it requires integration only over a subset of $\mathbb{T}$ (see (8.3)).

To be more precise, let Ψ_0 be a solution to the companion problem such that

$$\left\|\psi_0 + b\Psi_0 + bP_+\left(\bar{b}\left(f \vee h\right)\right) - h\right\|^2_{L^2(J)} = M^2 + \delta_{M^2},$$

where δ_{M^2} measures accuracy of the solution of the companion problem. Then, solution to the original problem should be sought as a series expansion near (6.1) with respect to δ_{M^2} as a small parameter

$$\tilde{g}_0 = \psi_0 - bP_+\left(\bar{b}\psi_0\right) + bP_+\left(\bar{b}\left(f \vee h\right)\right) + b\left.\frac{dg_0}{d\mu}\right|_{\mu=0}\left.\frac{d\mu}{dM_0^2}\right|_{M_0^2=M^2}\delta_{M^2} + \ldots, \tag{6.2}$$

and further the relations (5.5)-(5.6) followed by $\left.\dfrac{d\mu}{dM_0^2}\right|_{M_0^2=M^2} = \left.\left(\dfrac{dM_0^2}{d\mu}\right)^{-1}\right|_{\mu=0}$ should be employed (here g_0 is as in (3.11)). Recalling Section 2, we note that the first two terms realize a projection of ψ_0 onto $\left(bH^2\right)^{\perp_{H^2}}$ which will be simply ψ_0 if (4.1) was used as the arbitrary interpolant (see Remark 7).

If the companion problem was solved with good accuracy so that δ_{M^2} is small, linear order approximation in δ_{M^2} may be sufficient to recover the solution of the original problem. However, this connection between solution of two problems is valid for arbitrary values of δ_{M^2} if one considers infinite series in δ_{M^2}. This can be formalized with use of the Faà di Bruno formula which provides explicit form of the Taylor expansion for the function composition $g_0\left(\mu\left(M_0^2\right)\right)$ in terms of the derivatives $\left.\dfrac{d^k g_0}{\left(d\mu\right)^k}\right|_{\mu=0}$ and $\left.\dfrac{d^k \mu}{\left(dM_0^2\right)^k}\right|_{M_0^2=M^2}$ for any $k \in \mathbb{N}_+$. Applying the product

[4]Alternatively, one can take any $L^2\left(J\right)$ function that has norm M.

rule and expression (5.5) successively it can be shown that, after collection of terms at each differentiation, we have

$$\frac{d^k g_0}{(d\mu)^k} = (-1)^k \, k! \, (1+\mu\phi)^{-k} \, \phi^k \tilde{\xi} \quad \Rightarrow \quad \left.\frac{d^k g_0}{(d\mu)^k}\right|_{\mu=0} = (-1)^k \, k! \phi^k \tilde{\xi}_0,$$

where

$$\tilde{\xi} := P_+ \left(0 \vee \left(g_0 + \bar{b}\left(\psi_0 - h\right) + \Psi_0\right)\right), \quad \tilde{\xi}_0 := \phi\left(P_+\left(\bar{b}\left(f \vee h\right)\right) - P_+\left(\bar{b}\psi_0\right) - \Psi_0\right).$$

As far as computation of derivates of $\frac{d\mu}{dM_0^2}$ is concerned, complexity of the expressions grows and precise pattern seem to be hard to find especially since implicit differentiation has to be repeated every time resulting in successive appearance of extra factor $\frac{d\mu}{dM_0^2}$. Even though in practice one may look at the truncated Taylor expansion $M_0^2(\mu)$ and, since derivatives $\frac{dM_0^2}{d\mu}$ are readily computable, use reversion of the series to obtain power series expansion of μ in terms of M_0^2 (for reversion of series coefficient formula, see [28]) or, alternatively, employ the Lagrange inversion theorem that yields the inverse function $\mu\left(M_0^2\right)$ as an infinite series, in the latter case we would have to decide at which term the both series should be truncated so that to preserve desired accuracy at given order of δ_{M^2}. For small δ_{M^2}, only few terms are needed to give quite accurate connection between solution of the original and companion problems. Those can be precomputed manually or using computer algebra systems once and such calculations need not be repeated iteratively.

7 Stability results

The issue to be discussed here is linear stability of the solution (3.6) with respect to all physical components that the expression (3.11) involves explicitly and implicitly. In practice, functions f, h are typically obtained by interpolating discrete boundary data and hence may vary depending on interpolation method, measurement positions $\{z_j\}_{j=1}^N$ are usually known with a small error and pointwise data $\{\omega_j\}_{j=1}^N$ are necessarily subject to a certain noise. Therefore, we assume that boundary data f, h are slightly perturbed by $\delta_f \in L^2(I)$, $\delta_h \in L^2(J)$ and internal data $\{\omega_j\}_{j=1}^N$ with measurement positions $\{z_j\}_{j=1}^N$ by complex vectors $\boldsymbol{\delta_\omega}$, $\boldsymbol{\delta_z} \in \mathbb{C}^N$, respectively. Varying one of the quantities while the rest are kept fixed, we are going to estimate separately the linear effects of such perturbations on the solution $\tilde{g}_0 = \psi + bg_0$ to (3.6), denoting the induced deviations as $\delta_{\tilde{g}}$.

Proposition 11. *For $\mu > -1$, $f \in L^2(I) \setminus \mathcal{A}^{\psi,b}\big|_I$, $h \in L^2(J)$, and small enough data perturbations $\delta_f \in L^2(I)$, $\delta_h \in L^2(J)$, $\boldsymbol{\delta_\omega}$, $\boldsymbol{\delta_z} \in \mathbb{C}^N$, the following estimates hold:*

(1) $\quad \|\delta_{\tilde{g}}\|_{H^2} \leq m_1 \left(1 + \dfrac{m_1 M^2}{m_0 \|\xi\|_{H^2}^2}\right) \|\delta_f\|_{L^2(I)},$

(2) $\quad \|\delta_{\tilde{g}}\|_{H^2} \leq \left[(1 + m_1(1+\mu))\left(1 + \dfrac{m_1 M^2}{m_0 \|\xi\|_{H^2}^2}\right) - 1\right] \|\delta_h\|_{L^2(J)},$

(3) $\quad \|\delta_{\tilde{g}}\|_{H^2} \leq (1 + |\mu| m_1)\left(1 + \dfrac{m_1 M^2}{m_0 \|\xi\|_{H^2}^2}\right) \max_{j=1,\dots,N} \left\| \prod_{\substack{k=1\\k\neq j}}^N \dfrac{z - z_k}{z_j - z_k} \right\|_{H^2} \|\boldsymbol{\delta_\omega}\|_{l^1},$

(4) $\quad \|\delta_{\tilde{g}}\|_{H^2} \leq \left(1 + \dfrac{m_1 M^2}{m_0 \|\xi\|_{H^2}^2}\right)\left(C_\mu^{(1)} \|\delta_b\|_{H^\infty} + C_\mu^{(2)} \|\delta_\psi\|_{H^2}\right),$

where

$$\xi := P_+\left(0 \vee \left(g_0 + \bar{b}(\psi - h)\right)\right), \quad m_0 := \min\left\{(1+\mu)^{-1}, 1\right\}, \quad m_1 := \max\left\{(1+\mu)^{-1}, 1\right\}, \tag{7.1}$$

$C_\mu^{(1)} := m_1\left(\|f \vee h\|_{L^2(\mathbb{T})} + |\mu| \, \|h - \psi\|_{L^2(J)}\right), \quad C_\mu^{(2)} := 1 + |\mu| m_1, \qquad$ *and*

$$\|\delta_b\|_{H^\infty} \leq 2 \max_{j=1,\dots,N} \left\|(z - z_j)^{-1}\right\|_{H^\infty} \|\boldsymbol{\delta_z}\|_{l^1},$$

$$\|\delta_\psi\|_{H^2} \leq 2 \max_{j=1,\dots,N} |\omega_j| \max_{j=1,\dots,N} \left\| \prod_{\substack{m=1\\m\neq j}}^N (z - z_m) \right\|_{H^2} \times$$

$$\max_{j=1,\dots,N} \sum_{\substack{k=1\\k\neq j}}^N |z_j - z_k|^{-1} \left(\min_{j=1,\dots,N} \prod_{\substack{k=1\\k\neq j}}^N |z_j - z_k| \right)^{-1} \|\boldsymbol{\delta_z}\|_{l^1},$$

Proof. When the quantities entering the solution (3.11) vary, the overall variation of the solution δ_g will consist of parts entering the solution formula explicitly δ_{g_0} as well as those coming from the change of the norm of g_0 on J which leads to readjustment of the Lagrange parameter δ_μ so that the quantity $M_0^2(\mu) = \|\psi + bg_0(\mu) - h\|_{L^2(J)}^2$ be equal to the same given constraint M^2. For the sake of brevity, we are going to use the notations ξ, m_0 and m_1 introduced in (7.1) to denote certain quantities entering common estimates. The spectral bounds (3.13) for $\mu > -1$ imply

$$\sigma(1 + \mu\phi) \geq \min\{1 + \mu, 1\}, \quad \sigma(1 + \mu\phi) \leq \max\{1 + \mu, 1\}$$

$$\Rightarrow \quad \left\|(1+\mu\phi)^{-1}\right\| \leq \max\left\{(1+\mu)^{-1}, 1\right\}, \quad \left\|(1+\mu\phi)^{-1}\right\| \geq \min\left\{(1+\mu)^{-1}, 1\right\},$$

and so, in particular,

$$\mathrm{Re}\left\langle (1+\mu\phi)^{-1}\xi, \xi\right\rangle_{L^2(\mathbb{T})} \geq m_0 \|\xi\|_{H^2}^2 .$$

Then, the connection between δ_{M^2} denoting the change of $M_0^2(\mu)$ and δ_μ can be established based on the strict monotonicity (5.6) of $M_0(\mu)$ which allows the following estimate by inversion

$$\delta_\mu = \frac{\delta_{M^2}}{\left(M_0^2(\mu)\right)'} = -\frac{\delta_{M^2}}{2\mathrm{Re}\left\langle (1+\mu\phi)^{-1}\xi, \xi\right\rangle_{L^2(\mathbb{T})}} \quad \Rightarrow \quad |\delta_\mu| \leq \frac{|\delta_{M^2}|}{2m_0 \|\xi\|_{H^2}^2}. \tag{7.2}$$

Note that the bound in the right-hand side is finite due to the fact that $\|\xi\|_{H^2} > 0$ which holds unless $M_0(\mu) = 0$, the situation that was initially ruled out by Corollary 2. Discussion on *a priori* estimate of $\|\xi\|_{H^2}$ will be given in Remark 12.
Following this strategy, we embark on consecutive proof of the results *(1)-(4)*.
Result (1):
This is the simplest case, the variation of $M_0^2(\mu)$ is induced only by change of g_0. Namely,

$$\delta_{M^2} = 2\mathrm{Re}\left\langle \psi + bg_0(\mu) - h, b\delta_{g_0}(\mu)\right\rangle_{L^2(J)}, \tag{7.3}$$

where

$$\delta_{g_0} = (1+\mu\phi)^{-1} P_+\left(\bar{b}\delta_f \vee 0\right). \tag{7.4}$$

Application of the Cauchy-Schwarz inequality to (7.3) yields

$$|\delta_{M^2}| \quad \leq \quad 2M_0(\mu)\left\|(1+\mu\phi)^{-1}\right\| \left\|P_+\left(\bar{b}\delta_f \vee 0\right)\right\|_{L^2(\mathbb{T})} \leq 2M_0(\mu)\, m_1 \|\delta_f\|_{L^2(I)} .$$

and hence, by (7.2),

$$|\delta_\mu| \leq \frac{m_1 M_0(\mu)}{m_0 \|\xi\|_{H^2}^2} \|\delta_f\|_{L^2(I)} .$$

Now since $\delta_{\tilde{g}} = b\delta_g$, due to (5.5), we have

$$\delta_{\tilde{g}} = b\delta_{g_0} - b(1+\mu\phi)^{-1} P_+\left(0 \vee \left(g_0 + \bar{b}(\psi - h)\right)\right)\delta_\mu, \tag{7.5}$$

from where we deduce the inequality *(1)*.
Result (2):
This is totally analogous to the previous result except for now we have

$$\delta_{M^2} = 2\mathrm{Re}\left\langle \psi + bg_0(\mu) - h, b\delta_{g_0}(\mu) - \delta_h\right\rangle_{L^2(J)} \tag{7.6}$$

with

$$\delta_{g_0} = (1+\mu\phi)^{-1} P_+\left(0 \vee (1+\mu)\,\bar{b}\delta_h\right). \tag{7.7}$$

Therefore,

$$|\delta_{M^2}| \leq 2M_0(\mu)\left[1 + (1+\mu)\, m_1\right] \|\delta_h\|_{L^2(J)} \quad \Rightarrow \quad |\delta_\mu| \leq \frac{M_0(\mu)\left[1+(1+\mu)\, m_1\right]}{m_0 \|\xi\|_{H^2}^2} \|\delta_h\|_{L^2(J)} .$$

Feeding this in the relation (7.5), which still holds in this case, gives

$$\|\delta_{\tilde{g}}\|_{H^2} \leq m_1 \left(1 + \mu + \frac{[1 + (1+\mu)\, m_1]\, M^2}{m_0 \|\xi\|_{H^2}^2}\right) \|\delta_h\|_{L^2(J)},$$

that is exactly a rewording of estimate *(2)*.
Result (3):
The estimates *(3)* and *(4)* explore sensitivity of solution to measurement noise which any experimental data are prone to. In both cases proofs are similar to those of *(1)-(2)* with only few new ingredients.
In case of *(3)*, a perturbed data vector $\boldsymbol{\delta_\omega} \in \mathbb{C}^N$ affects the solution $\tilde{g}_0$ by means of the induced variation of ψ that we will denote by $\delta_\psi \in H^2(\mathbb{D})$.
If ψ is given by (4.1), its perturbation can be estimated as

$$\|\delta_\psi\|_{H^2} \leq \max_{k=1,\dots,N} \|\mathcal{K}(z_k,\cdot)\|_{H^2} \|S\|_1 \|\boldsymbol{\delta_\omega}\|_{l^1}, \tag{7.8}$$

where $\|\boldsymbol{\delta_\omega}\|_{l^1} := \sum_{k=1}^N |(\delta_\omega)_k|$, $\|S\|_1 := \max_{j=1,\dots,N} \sum_{k=1}^N |S_{kj}|$ with S as defined in (4.2). However, to get more explicit result with respect to data positions $\{z_j\}_{j=1}^N$ (which will be more relevant in case *(4)*) avoiding reference to (4.2), we employ polynomial interpolation in Lagrange form

$$\psi = \sum_{j=1}^N \omega_j \prod_{\substack{k=1\\k\neq j}}^N \frac{z - z_k}{z_j - z_k}, \tag{7.9}$$

in which case we have

$$\|\delta_\psi\|_{H^2} \leq \max_{j=1,\dots,N} \left\| \prod_{\substack{k=1\\k\neq j}}^N \frac{z - z_k}{z_j - z_k} \right\|_{H^2} \|\boldsymbol{\delta_\omega}\|_{l^1}. \tag{7.10}$$

Nevertheless, we note that the choice of interpolant (7.9) is not good for practical usage (making way for the barycentric interpolation formula, see [11]), but done only for the sake of analysis (again recall that, by Lemma 2, the final solution $\tilde{g}_0$ does not depend on a particular choice of the interpolant). In particular, we see that closedness of interpolation points amplifies the bound in the right-hand side which corresponds to ill-conditioning of the matrix $\mathcal{K}(z_k, z_j)$ for the choice of interpolant (4.1).
From this point on, we follow the same steps as in case *(2)* with (7.6)-(7.7) replaced by

$$\delta_{M^2} = 2\mathrm{Re}\,\langle \psi + bg_0(\mu) - h, \delta_{\tilde{g}_0}(\mu)\rangle_{L^2(J)}, \tag{7.11}$$

$$\delta_{\tilde{g}_0} = \delta_\psi - \mu (1 + \mu\phi)^{-1} P_+ \left(0 \vee \bar{b}\delta_\psi\right), \tag{7.12}$$

where the latter variation is estimated from (4.8). Then, we have

$$|\delta_{M^2}| \leq 2M_0(\mu)(1 + |\mu|\, m_1) \|\delta_\psi\|_{L^2(J)} \quad \Rightarrow \quad |\delta_\mu| \leq \frac{M_0(\mu)(1 + |\mu|\, m_1)}{m_0 \|\xi\|_{H^2}^2} \|\delta_\psi\|_{L^2(J)}. \tag{7.13}$$

Now

$$\delta_{\tilde{g}} = \delta_{\tilde{g}_0} - b (1 + \mu\phi)^{-1} P_+ \left(0 \vee \left(g_0 + \bar{b}(\psi - h)\right)\right) \delta_\mu, \tag{7.14}$$

and the resulting estimate *(3)* follows using (7.12)-(7.13) and recalling (7.10).

Result (4):
For a perturbation vector of positions $\boldsymbol{\delta_z} \in \mathbb{C}^N$, the respective deviation of the interpolant (7.9) is given by

$$\delta_\psi = \sum_{j=1}^{N} \omega_j \sum_{\substack{k=1 \\ k\neq j}}^{N} \left(\prod_{\substack{m=1 \\ m\neq k,j}}^{N} \frac{z - z_m}{z_j - z_m} \right) \frac{(z - z_j)(\delta_z)_k - (z - z_k)(\delta_z)_j}{(z_j - z_k)^2}, \tag{7.15}$$

and can be bounded, for instance, as

$$\|\delta_\psi\|_{H^2} \leq 2\omega_0 \max_{j=1,\dots,N} \left\| \prod_{\substack{m=1 \\ m\neq j}}^{N} (z - z_m) \right\|_{H^2} \frac{\max\limits_{j=1,\dots,N} \sum_{\substack{k=1 \\ k\neq j}}^{N} |z_j - z_k|^{-1}}{\min\limits_{j=1,\dots,N} \prod_{\substack{k=1 \\ k\neq j}}^{N} |z_j - z_k|} \|\boldsymbol{\delta_z}\|_{l^1}, \tag{7.16}$$

where $\omega_0 := \max\limits_{j=1,\dots,N} |\omega_j|$. However, more compact but even rougher bounds can be obtained in terms of d_0^{-N}, where $d_0 := \min\limits_{\substack{j,k=1,\dots,N \\ j\neq k}} |z_j - z_k|$, which are undesirable for large number of points that are not uniformly spaced.
This case is the most tedious one since now, in addition to ψ, the Blaschke products undergo the variation

$$\delta_b = \sum_{j=1}^{N} \left(\prod_{\substack{m=1 \\ m\neq j}}^{N} \frac{z - z_m}{1 - \bar{z}_m z} \right) \frac{z(z - z_j)(\delta_{\bar{z}})_j - (1 - \bar{z}_j z)(\delta_z)_j}{(1 - \bar{z}_j z)^2}, \tag{7.17}$$

which can be estimated as

$$\begin{aligned} \|\delta_b\|_{H^\infty} &\leq \max_{j=1,\dots,N} \left(\left\| \frac{z(z - z_j)}{(1 - \bar{z}_j z)^2} \right\|_{H^\infty} + \left\| (1 - \bar{z}_j z)^{-1} \right\|_{H^\infty} \right) \|\boldsymbol{\delta_z}\|_{l^1} \\ &= 2 \max_{j=1,\dots,N} \left\| (z - z_j)^{-1} \right\|_{H^\infty} \|\boldsymbol{\delta_z}\|_{l^1}. \end{aligned} \tag{7.18}$$

The rest of the computations is most similar to those in case *(3)* but slightly more general. Namely, (7.11) and (7.14) hold with

$$\begin{aligned} \delta_{\tilde{g}_0} &= \delta_\psi + \delta_b (1 + \mu\phi)^{-1} \left[P_+ \left(\bar{b}(f \vee h) \right) + \mu P_+ \left(0 \vee \bar{b}(h - \psi) \right) \right] \\ &\quad + b(1 + \mu\phi)^{-1} \left[P_+ (\delta_{\bar{b}} (f \vee h)) + \mu P_+ (0 \vee \delta_{\bar{b}} (h - \psi)) \right] - \mu b (1 + \mu\phi)^{-1} P_+ \left(0 \vee \bar{b}\delta_\psi \right) \end{aligned}$$

estimated from (4.8). Therefore,

$$\|\delta_{\tilde{g}}\|_{H^2} \leq m_1 \left(1 + \frac{m_1 M^2}{m_0 \|\xi\|_{H^2}^2} \right) \|\delta_{\tilde{g}_0}\|_{H^2},$$

$$\|\delta_{\tilde{g}_0}\|_{H^2} \leq (1 + |\mu| m_1) \|\delta_\psi\|_{H^2} + m_1 \left(\|f \vee h\|_{L^2(\mathbb{T})} + |\mu| \|h - \psi\|_{L^2(J)} \right) \|\delta_b\|_{H^\infty},$$

and the final estimate *(4)* follows. □

Remark 12. The quantity ξ introduced in (7.1) enters the results *(1)*-*(4)* and should be bounded away from zero. This fact, however, follows from Proposition 2 and Corollary 2. Moreover, the norm of ξ can be *a priori* estimated as

$$\|\xi\|_{H^2} \geq \frac{1}{|\mu|} \left(M - \left\| \psi - h + bP_+ \left(\bar{b}(f \vee h) \right) \right\|_{L^2(J)} \right) \tag{7.19}$$

by applying the triangle inequality for $L^2(J)$ norm of the quantity

$$\psi + bg_0 - h = \psi - h + bP_+\left(\bar{b}\,(f \vee h)\right) + \mu b P_+\left(0 \vee \left(\bar{b}\,(h-\psi) - g_0\right)\right),$$

which is a consequence of (3.15). Of course, the estimate (7.19) is useful only under assumption

$$\left\|\psi - h + bP_+\left(\bar{b}\,(f \vee h)\right)\right\|_{L^2(J)} < M, \tag{7.20}$$

but we do not include it in formulation of the Proposition, since this inequality can be achieved without imposing any restriction on given boundary data f and h or increasing the bound M: since, according to Lemma 2, choice of ψ does not affect solution $\tilde{g}_0$ whose stability we are investigating, one can consider another instance of bounded extremal problem, now formulated for $\psi \in H^2(\mathbb{D})$ meeting pointwise constraints (3.1) and approximating $h - bP_+\left(\bar{b}\,(f \vee h)\right) \in L^2(J)$ on J sufficiently closely (with precision M) with a finite bound on I without any additional information (meaning that for such a problem "I" $= J$, "h" $= 0$). To be more precise, given arbitrary $\psi_0 \in H^2(\mathbb{D})$ satisfying pointwise interpolation conditions (3.1) (for instance, one can use (4.1)), we represent $\psi = \psi_0 + b\Psi$ and thus search for approximant $\Psi \in H^2(\mathbb{D})$ to "f" $= \bar{b}\,(h - \psi_0) - P_+\left(\bar{b}\,(f \vee h)\right) \in L^2(J)$ such that $\|\Psi\|_{L^2(I)} = \tilde{M}$ for arbitrary $\tilde{M} \in (0, \infty)$. We also note that in the case of reduction to the previously considered problem with no pointwise data imposed ([5], [6]), i.e. when $\psi \equiv 0$, $b \equiv 1$, one does not have flexibility of varying the interpolant. However, the stability estimates still persist in the region of interest (that is, for $-1 < \mu < 0$) since the condition (7.20) is fulfilled as long as $\mu < 0$ due to (3.11) evaluated at $\mu = 0$ and (5.3).

Remark 13. Results *(3)-(4)* technically show stability in terms of finite pointwise data sets $\{\omega_j\}_{j=1}^N$, $\{z_j\}_{j=1}^N$ in l^1 norm, however, by the equivalence of norms in finite dimensions, the same results, but with different bounds, also hold for l^p norms, for any $p \in \mathbb{N}_+$ and $p = \infty$.

8 Numerical illustrations and algorithmic aspects

To illustrate the results of Sections 4-5 and estimate practical computational parameters, we perform the following numerical simulations. First of all, without loss of generality, choose $J = \left\{e^{i\theta} : \theta \in [-\theta_0, \theta_0]\right\}$ for some fixed $\theta_0 \in (0, \pi)$. In order to invert the Toeplitz operator in (3.11) in a computationally efficient way, we realize projection of equation (3.15) onto finite dimensional (truncated) Fourier basis $\left\{z^{k-1}\right\}_{k=1}^{Q}$ for large enough $Q \in \mathbb{N}_+$ and look for approximate solution in the form

$$g(z) = \sum_{k=1}^{Q} g_k z^{k-1}. \tag{8.1}$$

Introducing, for $m,\, k \in \{1, \ldots, Q\}$,

$$A_{k,m} := \begin{cases} \dfrac{\sin(m-k)\,\theta_0}{\pi\,(m-k)}, & m \neq k, \\ \theta_0/\pi, & m = k, \end{cases} \qquad A := [A_{k,m}]_{k,m=1}^{Q}\,, \tag{8.2}$$

$$s_k := \left\langle \left(\bar{b}\,(f-\psi) \vee (1+\mu)\,\bar{b}\,(h-\psi)\right), e^{i(k-1)\theta} \right\rangle_{L^2(0,2\pi)}, \qquad \boldsymbol{s} := [s_k]_{k=1}^{Q}\,, \tag{8.3}$$

the projection equation

$$\left\langle (1+\mu\phi)\,g - P_+\left(\bar{b}\,(f-\psi) \vee (1+\mu)\,\bar{b}\,(h-\psi)\right), z^{k-1} \right\rangle_{L^2(\mathbb{T})} = 0$$

becomes the vector equation (if we employ 1 to denote the identity $Q \times Q$ matrix)

$$(1+\mu A)\,\boldsymbol{g} = \boldsymbol{s}, \qquad \boldsymbol{g} := [g_k]_{k=1}^{Q}$$

with a real symmetric Toeplitz matrix which is computationally cheap to invert: depending on the algorithm, asymptotic complexity of inversion may be as low as $\mathcal{O}\left(Q \log^2 Q\right)$ (see [12] and references therein).

Now, in order to numerically demonstrate the monotonicity results (5.3) for e and M_0 with respect to the parameter μ and to compare the behavior with that of series expansions (5.18)-(5.19), we run simulation for the following set of data. We choose $N = 5$, $\theta_0 = \pi/3$, and

$$f(\theta) = f_0(\theta) + \frac{\epsilon}{\exp(i\theta) - 0.4 - 0.3i}, \qquad f_0(\theta) := \exp(5i\theta) + \exp(2i\theta) + 1 \in \mathcal{A}^{\psi,b}$$

(when the parameter $\epsilon \neq 0$, obviously, $f \in L^2(I)$ does not extend inside the disk as a H^2 function). Further, f_0 is the restriction of the function $z^5 + z^2 + 1$ satisfying pointwise interpolation conditions (3.1) for points $\{z_j\}_{j=1}^{5}$ and values $\{\omega_j\}_{j=1}^{5}$ chosen as given in Table 1. We also take $h \in L^2(J)$ as

$$h(\theta) = \frac{1}{\exp(i\theta) - 0.5i}.$$

Based on the points $\{z_j\}_{j=1}^{5}$, we construct the Blaschke product according to (2.8) with the choice of constant $\phi_0 = 0$ (obviously, final physical results should not depend on a choice of this auxiliary parameter which is also clear from the solution formula (4.8)). The interpolant ψ was chosen as (4.1). Series expansions (5.18)-(5.19) are straightforward to evaluate numerically since $F(k)$ involves the quantity ξ_0 given by (5.17). The projections P_+ there are computed by performing non-negative-power expansions as (8.1) whereas ϕ^k is simply iterative multiplication

of the first Q Fourier coefficients of ξ_0 by the Toeplitz operator matrix (8.2). Such iterations are extremely cheap to compute once the matrix A is diagonalized.
Figures 8.1-8.2 illustrate approximation errors on I and discrepancies on J versus the parameter μ for different values of ϵ when the dimension of the solution space is fixed to $Q = 20$. Number of terms in the series expansions (5.18)-(5.19) was kept fixed at $S = 10$ (such that it is the maximal power of μ in the series). It is remarkable that even such a low number of terms gives bounds which are in very reasonable agreement with those computed from solution up to relatively close neighborhood of $\mu = -1$. On Figure 8.3, we further investigate change of deviation of the series expansion from the solution computed numerically (which is taken as a reference in this case, see the discussion in the next paragraph) as more terms are taken into account in the expansions.
Figure 8.4 shows variation of the results with respect to truncation of the solution basis while the parameter $\epsilon = 0.5$ is kept fixed. Errors are compared to results obtained for $Q = 50$ which is taken as reference. We conclude that a choice of Q between 10 and 20 is already sufficiently good for practical purposes. In particular, we can regard the numerical computation results obtained for $Q = 20$ as those corresponding to faithful solution so to compare them with what follows from the series expansions (5.18)-(5.19). Clearly, a choice of $Q < N = 5$ does not make sense since, according to Lemma 2, the interpolant ψ can be chosen as a polynomial which, under such a restriction, will not even be able to meet all pointwise constraints.
Finally, on Figure 8.5, we plot auxiliary quantities $F(k)$ and $kF(k)$ versus k which fundamentally enter the series expansions (5.18)-(5.19). In such a computation of multiple iterative action of the Toeplitz operator ϕ on a fixed H^2 function mentioned above, we used high value of $Q = 50$ to prevent possible accumulation of error steming from the truncation to a finite dimensional basis. The first quantity $F(k)$ demonstates the expected decay to zero, while the second one shows that the decay is not fast enough to produce a summable series (that is, $F(k) \neq o(1/k)$ as $k \to \infty$) which illustrates the sharpness of Lemma 3 and, on the other hand, is consistent with blow-up of $M_0^2(\mu)$ near $\mu = -1$.

Suggested computational algorithm Even though Figure 8.3 shows good accuracy of approximation $e(\mu)$ and $M_0^2(\mu)$ from the series expansions (5.18)-(5.19), it is clear, by nature of such expansions, that the convergence slows down as μ gets closer to -1, and hence, for the genuine values, the number of terms in the series should be increased dramatically. However, as it was mentioned, the quantities $F(k)$ are very cheap to compute. It remains only to estimate S, that is the number of terms in series for the accurate approximation of $e(\mu)$ and $M_0^2(\mu)$, but it suffices to perform such a calibration only once, namely, for the lowest value of μ in the computational range. This suggests the following computational strategy:
1. Decide on the lowest value of the Lagrange parameter μ_0 by checking the approximation rate computed from solving the system (8). The quantity $e(\mu_0)$ will then be the best approximation rate on I.
2. Determine the number of terms S by comparing the approximation rate with that evaluated from the expansion (5.19) for μ_0.
3. Fix S, precompute the values $F(k)$, $k = 1, \ldots, S$. Vary the parameter μ and evaluate the approximation and blow-up rates from the expansions (5.18)-(5.19) in order to find a suitable trade-off.

z	ω
$0.5+0.4i$	$0.9852+0.3752i$
$-0.3+0.3i$	$1.0097-0.1897i$
$0.2+0.6i$	$0.7811+0.2362i$
$0.2-0.5i$	$0.8328-0.1852i$
$0.8-0.1i$	$1.9069-0.3584i$

Table 1: Interior pointwise data

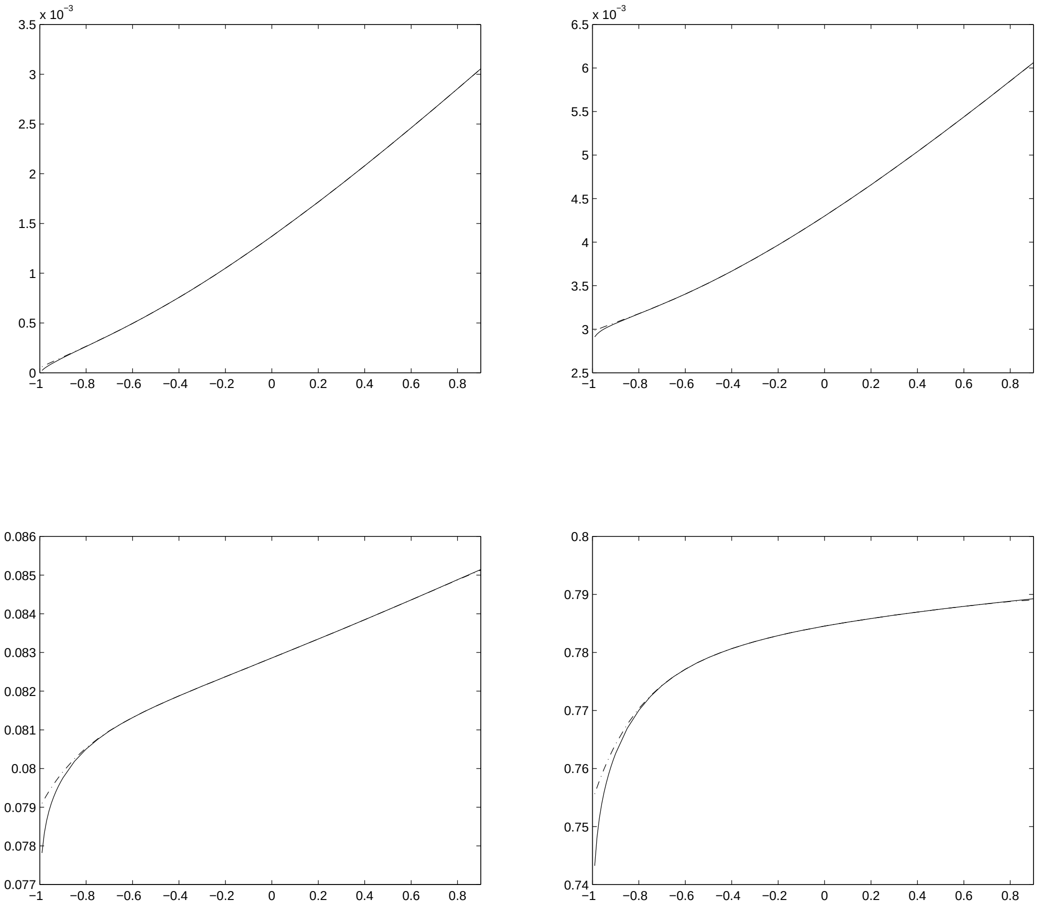

Figure 8.1: Relative approximation error on I: $e(\mu)/\|f\|_{L^2(I)}$ from solution (solid) and series expansion (dash-dot) for ϵ=0 (top left), $\epsilon=0.1$ (top right), $\epsilon=0.5$ (bottom left), $\epsilon=2$ (bottom right).

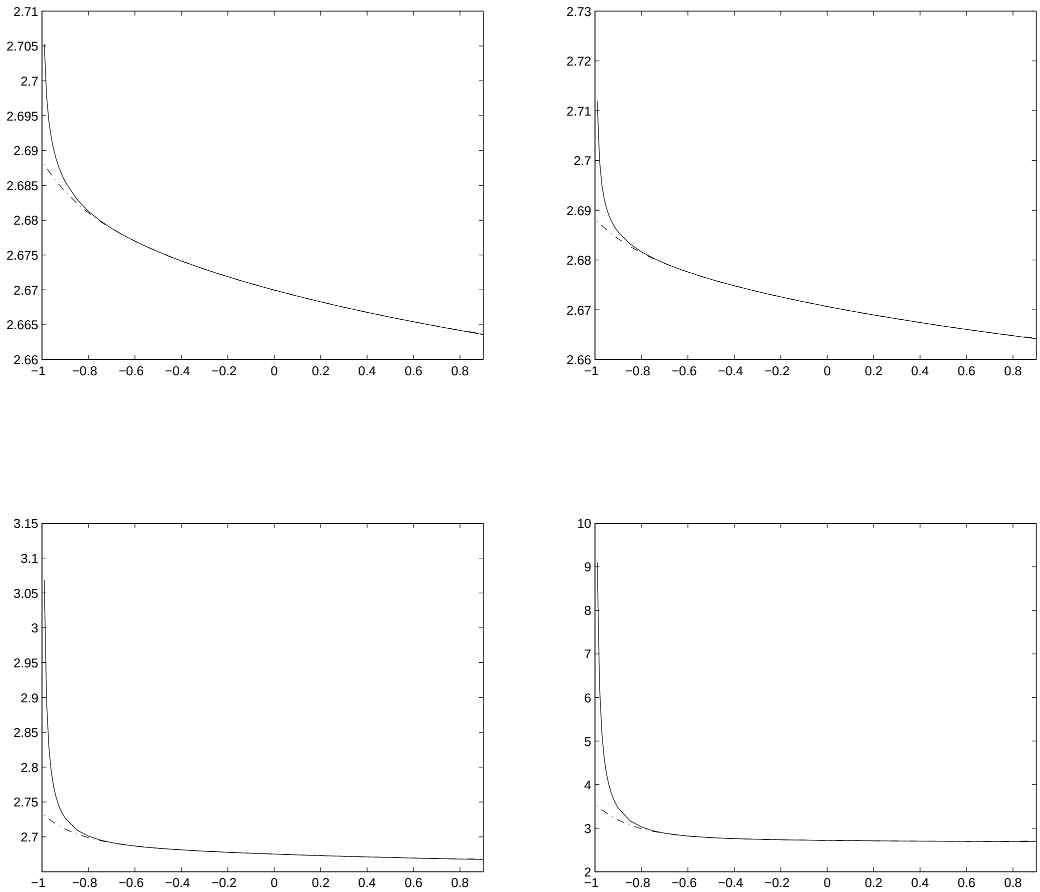

Figure 8.2: Relative discrepancy on J: $M_0^2(\mu) \,/\, \|h\|_{L^2(J)}$ from solution (solid) and series expansion (dash-dot) for ϵ=0 (top left), $\epsilon = 0.1$ (top right), $\epsilon = 0.5$ (bottom left), $\epsilon = 2$ (bottom right).

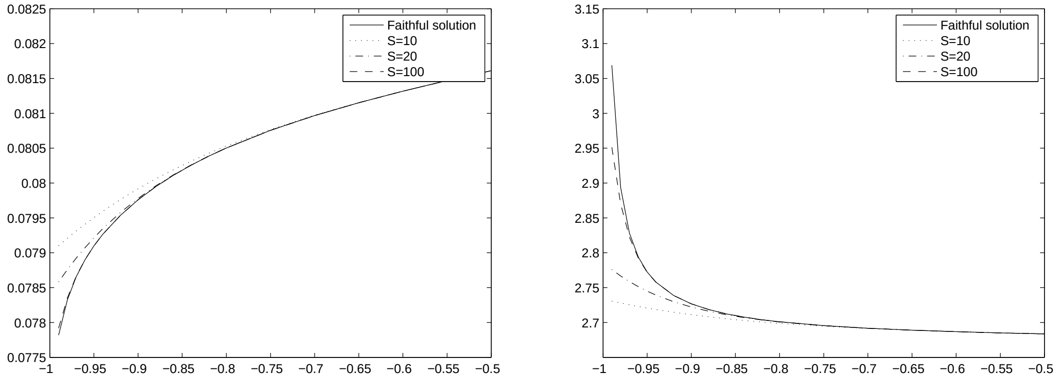

Figure 8.3: Relative approximation error on I (left) and relative discrepancy error on J (right).

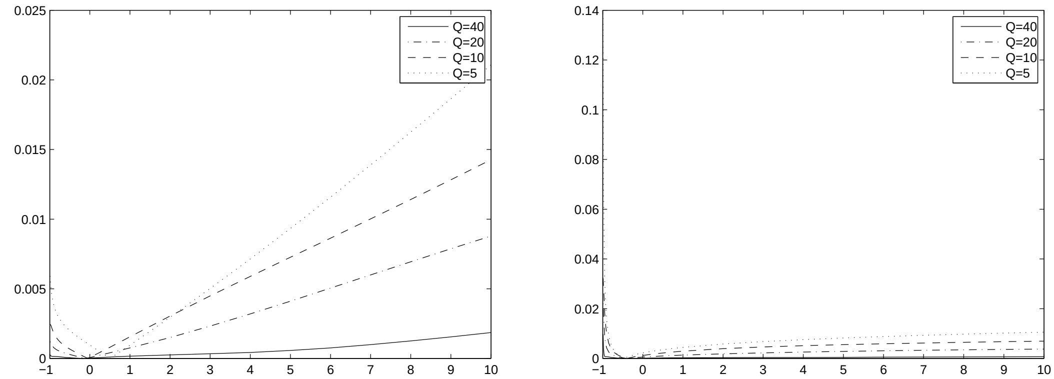

Figure 8.4: Errors on I (left) and J (right) compared to results for $Q = 50$.

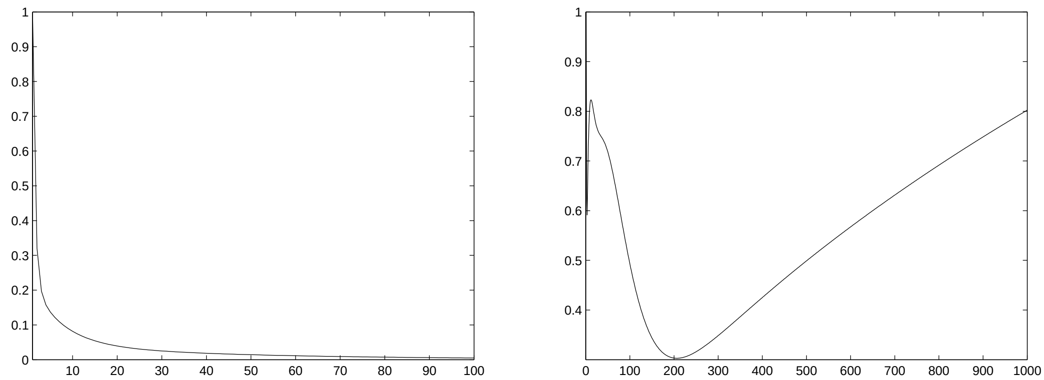

Figure 8.5: Auxiliary quantities $F(k)$ and $kF(k)$ computed with $Q = 50$.

9 Conclusions

Motivated by some physical applications, we have introduced and solved a bounded extremal problem that extends the one of best norm-constrained approximation of a given function on a subset of the circle by the trace of a H^2 function [6] to the case where additional pointwise constraints are imposed inside the unit disk.
Under such a formulation, there were obtained new results which apply to a problem without pointwise constraints, as a particular case. Namely, we suggested a method of computing the approximation rate and the discrepancy growth in terms of a Lagrange parameter. With an extra argument, the method was used to deduce asymptotic estimates for quantities governing the approximation quality relying on a different approach as compared to [7] and possible extensions of those results were discussed. The new series expansion method was further numerically demonstrated to be very efficient especially beyond the asymptotic regime, thus making redundant solving multiple instances of the bounded extremal problem iteratively aiming to find the Lagrange parameter value corresponding to a suitable trade-off between approximation rate and control of the blow-up.
We have also observed a connection to a companion problem which is intrinsic to the presence of internal pointwise data in the original one. Solution of such a companion problem is computationally cheaper which may be of big advantage when solving multiple instances of the original bounded extremal problem. However, there is still room for further investigations in this direction.
Another gap that was filled with the present work is stability estimates for bounded extremal problems with fixed constraints. Even without presence of pointwise data, the only available result, to our knowledge, is a proof of continuity of the solution with respect to approximated function without additional data ($h = 0$; see [9, Sect. 4.3.4]).
Since the considered formulation is rather general and has potentially many physical applications, there are number of issues one may further want to look into. For example, it would be interesting to see how the choice of positions of pointwise interpolation data affects the solution. How does increasing the number of points boost the approximation rate and lower the discrepancy growth significantly? With the same quantity N of pointwise constraints, are the results better when points are located closer to the boundary, when they are spread out evenly in the disk or concentrated in an area or put along a curve? Physically, if positions of sensors from which the boundary data are obtained are not precise, does it worth to single out some far out points to be excluded from interpolation of boundary data functions in order to be treated as internal constraints? Though some insights into these questions can be obtained numerically from already developed software, the precise analysis of some issues is expected to be quite involved.
Another extension of the results may be considered in direction of generalized analytic functions and annular domains [17, 22].

APPENDIX

Theorem. *(Hartman-Wintner)*
Let $\xi \in L^\infty(\mathbb{T})$: $\mathbb{T} \to \mathbb{R}$ be a symbol defining the Toeplitz operator $T_\xi : H^2 \to H^2 : F \mapsto T_\xi(F) = P_+(\xi F)$. Then, the operator spectrum is $\sigma(T_\xi) = [\text{ess inf}\, \xi,\ \text{ess sup}\, \xi] \subset \mathbb{R}$.

Proof. We give a proof combining ideas from both [14, Th. 7.20] and [29, Th. 4.2.7] in a way such that it is short and self-consistent.
First of all, since ξ is a real-valued function, T_ξ is self-adjoint, and hence $\sigma(T_\xi) \subset \mathbb{R}$.
Now, to prove the result, we employ definition of $\sigma(T_\xi)$ as complement of resolvent set, namely, given $\mu \in \mathbb{R}$, we aim to show that the existence and boundedness of $(T_\xi - \mu I)^{-1}$ on H^2 (i.e. when μ is in the resolvent set) necessarily imply that either $\xi - \mu > 0$ or $\xi - \mu < 0$ a.e., in other words, $(\xi - \mu)$ must be strictly uniform in sign a.e. on $\mathbb{D}$.
Assume μ is fixed so that the inverse of $(T_\xi - \mu I)$ exists and bounded on the whole H^2, in particular, on constant functions. This means that there is $f \in H^2$ such that

$$T_{\xi-\mu} f = (T_\xi - \mu I) f = 1.$$

For any $n \in \mathbb{N}_+$, denoting f_k the coefficients of Fourier expansion of f on $\mathbb{T}$, let us evaluate

$$\langle T_{\xi-\mu} f, z^n f \rangle_{L^2(\mathbb{T})} = \langle 1, z^n f \rangle_{L^2(\mathbb{T})} = \left\langle z^n, \bar{f} \right\rangle_{L^2(\mathbb{T})} = \sum_{k=0}^{\infty} f_k \int_0^{2\pi} e^{i(n+k)\theta} d\theta = 0.$$

On the other hand, since $z^n f \in H^2$, we have

$$\langle T_{\xi-\mu} f, z^n f \rangle_{L^2(\mathbb{T})} = \langle (\xi - \mu) f, z^n f \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} (\xi - \mu) |f|^2 \bar{z}^n d\sigma,$$

and thus

$$\int_{\mathbb{T}} (\xi - \mu) |f|^2 z^{-n} d\sigma = 0, \quad n \in \mathbb{N}_+,$$

which implies that $(\xi - \mu)|f|^2$ cannot be an analytic function on $\mathbb{D}$ unless it is constant.
However, since ξ and μ are real-valued, taking conjugation yields

$$\int_{\mathbb{T}} (\xi - \mu) |f|^2 z^{n} d\sigma = 0, \quad n \in \mathbb{N}_+,$$

which prohibits $(\xi - \mu)|f|^2$ being non-analytic on $\mathbb{D}$ either. Therefore, $(\xi - \mu)|f|^2 = \text{const}$, and hence $(\xi - \mu)$ has constant sign a.e. on $\mathbb{D}$ that proves the result. □

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RESEARCH CENTRE
SOPHIA ANTIPOLIS – MÉDITERRANÉE

2004 route des Lucioles - BP 93
06902 Sophia Antipolis Cedex

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Inria
Domaine de Voluceau - Rocquencourt
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