# ON THE LIMITING AMPLITUDE PRINCIPLE FOR THE WAVE EQUATION WITH VARIABLE COEFFICIENTS

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Abstract. In this paper, we prove new results on the validity of the limiting amplitude principle (LAP) for the wave equation with nonconstant coefficients, not necessarily in divergence form. Under suitable assumptions on the coefficients and on the source term, we establish the LAP for space dimensions 2 and 3. This result is extended to one space dimension with an appropriate modification. We also quantify the LAP and thus provide estimates for the convergence of the time-domain solution to the frequency-domain solution. Our proofs are based on time-decay results of solutions of some auxiliary problems.

**Keywords** wave equation, variable coefficients, limiting amplitude principle, long-time asymptotics

**Mathematics Subject Classification** 35L05, 35L10, 35B10, 35B40

## 1. Introduction

An essential ingredient in connecting time- and frequency-domain wave problems is the limiting amplitude principle (LAP). Originally proposed as one of the tools to select the unique solution of the Helmholtz equation problem in an infinite domain, it has been studied in numerous works over the last 70 years.

The LAP can be crudely stated as follows: The solution to the time-dependent wave equation with time-harmonic source term converges, for large times, to the solution of the Helmholtz equation with the spatial source term and frequency corresponding to the original time-harmonic source. Our main motivation for revisiting the LAP comes from numerical analysis. Helmholtz problems can be challenging to solve in practice for large wavenumbers. Numerical methods have been proposed to address a classical Helmholtz problem efficiently through its reformulations in the time domain. They include the controllability method introduced in [7, 17], together with its spectral version [22] and its extensions [21, 20], the WaveHoltz method [1], the time-domain preconditioner of [32], and the front-tracking adaptive method of [2]. Numerical results presented in the latter paper, for instance, have shown that solving the Helmholtz equation in the time domain can be advantageous for high frequencies, when computations are essentially reduced to the neighborhood of a lower-dimensional manifold (wave-front area).

The analysis of these methods requires a quantification of the modeling error (reformulation of the frequency-domain problem into a time-domain problem), which will add to the error due to the numerical approximation of the problem in the time domain. This motivates the study of the LAP under some new angles, with particular focus on the quantification of large-time convergence.

As opposed to a direct study of the resolvent operator, our analysis is based on decay estimates for the solutions of some auxiliary PDE problems. Since decay results are still the subject of intense investigation, an advantage of this approach is that new findings in that area directly translate into improvements in the quantification of the large-time convergence in the LAP.

**Main results.** We consider the following setup. Given an angular frequency $\omega > 0$, material parameters $\alpha$, $\beta$, which smoothly vary within some bounded domain, and a compactly supported source term $F$, we consider the following frequency-domain and time-domain problems, respectively:

$$\begin{cases} -\omega^2 U(\mathbf{x}) - \beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla U(\mathbf{x})) = F(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d, \\ \lim\limits_{|\mathbf{x}| \to \infty} |\mathbf{x}|^{\frac{d-1}{2}} \left[ \partial_{|\mathbf{x}|} U(\mathbf{x}) - i\omega \sqrt{\beta_0/\alpha_0}\, U(\mathbf{x}) \right] = 0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \partial_t^2 u(\mathbf{x},t) - \beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla u(\mathbf{x},t)) = e^{-i\omega t} F(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d,\ t > 0, \\ u(\mathbf{x},0) = 0, \quad \partial_t u(\mathbf{x},0) = 0, & \mathbf{x} \in \mathbb{R}^d. \end{cases} \tag{1.2}$$

Our assumptions on $\alpha$, $\beta$, and $F$ are stated as follows.

**Assumption 1.1. (smoothness, compactly supported derivatives & positivity of coefficients)** *Assume $d \geq 2$, and let $\alpha,\ \beta \in C^\infty(\mathbb{R}^d)$ be real-valued functions such that $\alpha(\mathbf{x}) \geq \alpha_{\min}$, $\beta(\mathbf{x}) \geq \beta_{\min}$ for $\mathbf{x} \in \mathbb{R}^d$, and $\alpha(\mathbf{x}) \equiv \alpha_0$, $\beta(\mathbf{x}) \equiv \beta_0$ for $\mathbf{x} \in \mathbb{R}^d \backslash \Omega_{in}$, with some bounded domain $\Omega_{in} \subset \mathbb{R}^d$ and constants $\alpha_{\min}$, $\beta_{\min}$, $\alpha_0$, $\beta_0 > 0$.*

**Assumption 1.1'. (regularity, compactly supported derivatives & positivity of coefficients; $1D$ case)** *Assume $d = 1$, and let $\alpha,\ \beta \in W^{1,\infty}(\mathbb{R})$ be real-valued functions such that $\alpha(x) \geq \alpha_{\min}$, $\beta(x) \geq \beta_{\min}$ for $x \in \mathbb{R}$, and $\alpha(x) \equiv \alpha_0$, $\beta(x) \equiv \beta_0$ for $x \in \mathbb{R} \backslash \Omega_{in}$, with some open bounded interval $\Omega_{in} \subset \mathbb{R}$ and constants $\alpha_{\min}$, $\beta_{\min}$, $\alpha_0$, $\beta_0 > 0$.*

**Assumption 1.2. (nontrapping coefficients)** *Let $\alpha$, $\beta$ be non-trapping, i.e. such that all rays associated with the metric $\alpha/\beta$ escape to infinity* [5, Sect. 1]*. In other words (see e.g.* [18, Def. 7.6 & Cor. 7.10]*), defining $H(\mathbf{q},\mathbf{p}) := \alpha(\mathbf{q}) |\mathbf{p}|^2 - \beta(\mathbf{q})$, given $\mathbf{q}_0$, $\mathbf{p_0} \in \mathbb{R}^d$ such that $H(\mathbf{q_0},\mathbf{p_0}) = 0$, the solution vector of the canonical system of differential equations with the Hamiltonian $H(\mathbf{q},\mathbf{p})$,*

$$\begin{cases} \frac{d}{dt}\mathbf{q}(t) = 2\alpha(\mathbf{q})\,\mathbf{p}(t), & t > 0, \\ \frac{d}{dt}\mathbf{p}(t) = -\frac{\beta(\mathbf{q})}{\alpha(\mathbf{q})} \nabla_{\mathbf{q}} \alpha(\mathbf{q}) + \nabla_{\mathbf{q}} \beta(\mathbf{q}), & t > 0, \\ \mathbf{q}(0) = \mathbf{q}_0, \quad \mathbf{p}(0) = \mathbf{p}_0, \end{cases} \tag{1.3}$$

*must satisfy* $|\mathbf{q}(t)| \to \infty$ *as* $t \to \infty$.

**Assumption 1.3. (compactly supported source)** *Let the complex-valued function* $F \in L^2\left(\mathbb{R}^d\right)$ *be such that* $\operatorname{supp} F \subset \Omega_{in}$*, with* $\Omega_{in}$ *as in Assumption 1.1 or 1.1'.*

Under the above assumptions, we prove the following versions of the LAP.

**Theorem 1.4.** *Let* $d = 2, 3$. *Suppose that Assumptions 1.1–1.3 are satisfied. Let* $U(\mathbf{x})$ *and* $u(\mathbf{x}, t)$ *be solutions to* (1.1) *and* (1.2)*, respectively. Then, there exists a constant* $C > 0$ *depending on* $F$, $\alpha$, $\beta$, $\omega$, *and* $\Omega$ *such that*
*for* $d = 2$:

$$\left\|u(\cdot, t) - e^{-i\omega t}U\right\|_{H^1(\Omega)} + \left\|\partial_t u(\cdot, t) + i\omega e^{-i\omega t}U\right\|_{L^2(\Omega)} \le C\,\frac{1 + \log\left(1 + t^2\right)}{\left(1 + t^2\right)^{1/2}}, \quad t \ge 0;$$

*for* $d = 3$:

$$\left\|u(\cdot, t) - e^{-i\omega t}U\right\|_{H^1(\Omega)} + \left\|\partial_t u(\cdot, t) + i\omega e^{-i\omega t}U\right\|_{L^2(\Omega)} \le \frac{C}{\left(1 + t^2\right)^{1/2}}, \quad t \ge 0,$$

*where* $\Omega \subset \mathbb{R}^d$ *is an arbitrary bounded domain.*

**Theorem 1.5.** *Let* $d = 1$. *Suppose that Assumptions 1.1' and 1.3 are satisfied. Let* $U(\mathbf{x})$ *and* $u(\mathbf{x}, t)$ *be solutions to* (1.1) *and* (1.2)*, respectively. Then, there exist constants* $\Lambda > 0$ *(depending on* $\alpha$, $\beta$*) and* $C > 0$ *(depending on* $F$, $\alpha$, $\beta$, $\omega$, *and* $\Omega$*) such that*

$$\left\|u(\cdot, t) - e^{-i\omega t}U - U_\infty\right\|_{H^1(\Omega)} + \left\|\partial_t u(\cdot, t) + i\omega e^{-i\omega t}U\right\|_{L^2(\Omega)} \le Ce^{-\Lambda t}, \quad t \ge 0,$$

*where*

$$U_\infty := \frac{1}{2i\omega\sqrt{\alpha_0\beta_0}} \int_{\Omega_{in}} F(x)\,\beta(x)\,dx, \tag{1.4}$$

*and* $\Omega \subset \mathbb{R}^d$ *is an arbitrary bounded domain.*

**Remark 1.6.** *Note that, in contrast with Theorem 1.4, Theorem 1.5 does not require Assumption 1.2. This is because in the one-dimensional setting, the rays can only be associated with left- and right-propagating waves, and the trapping cannot occur for the regular coefficients.*

These results are summarized in Table 1.

**Previous results on the LAP.** Let us provide a brief overview of previous works on the LAP. The simplest version of the LAP dealing with the constant-coefficient, three-dimensional wave equation has been known at least since 1948 [34, 35]. There, it is proven that this physical principle selects the unique solution of the stationary problem satisfying the Sommerfeld radiation condition.

Starting from the seminal work by Ladyzhenskaya [24], variable-coefficient equations of the form $\partial_t^2 u(\mathbf{x}, t) - c^2(\mathbf{x})\,\Delta u(\mathbf{x}, t) + q(\mathbf{x})\,u(\mathbf{x}, t) = f(\mathbf{x})\,e^{-i\omega t}$ have been considered. Namely, while [24, 26] treat the case $c \equiv 0$, paper [27] deals with the case $q \equiv 0$. When $q$, $\nabla c$ and $f$ are sufficiently localised, the validity of the LAP is proven in a pointwise sense but a rate of the convergence is not specified.

| | $d=1$ | $d=2$ | $d=3$ |
|---|---|---|---|
| assumptions | 1.1', 1.3 | 1.1, 1.2, 1.3 | 1.1, 1.2, 1.3 |
| $u^{\text{DIFF}}(\mathbf{x},t)$ | $u^{\text{W}}(\mathbf{x},t)-u^{\text{H}}(\mathbf{x},t)-U_\infty$ | $u^{\text{W}}(\mathbf{x},t)-u^{\text{H}}(\mathbf{x},t)$ | $u^{\text{W}}(\mathbf{x},t)-u^{\text{H}}(\mathbf{x},t)$ |
| bound of $\left\|u^{\text{DIFF}}(\cdot,t)\right\|_\star$ | $Ce^{-\Lambda t}$ | $C\,\frac{1+\log(1+t^2)}{(1+t^2)^{1/2}}$ | $C\,\frac{1}{(1+t^2)^{1/2}}$ |
| statement | Thm. 1.5 | Thm. 1.4 | Thm. 1.4 |
| proof | Sect. 3 | Sect. 3 | Sect. 3 |
| time-decay results used | Prop. 2.5 (see [3, Thm 1.4]) | Prop. 2.1, Prop. 2.2 Lem. 2.3, Lem. 2.4 (proofs: Sect. 4) | Prop. 2.1, Prop. 2.2 Lem. 2.3, Lem. 2.4 (proofs: Sect. 4) |

TABLE 1. Summary of our results. Here, $u^{\text{W}}(\mathbf{x},t)$ is the solution to the wave problem (1.2), $u^{\text{H}}(\mathbf{x},t) := e^{-i\omega t}U(\mathbf{x})$, with $U$ solution to the Helmholtz problem (1.1), $U_\infty$ is the constant in (1.4), and $\left\|u^{\text{DIFF}}(\cdot,t)\right\|_\star := \left\|u^{\text{DIFF}}(\cdot,t)\right\|_{H^1(\Omega)} + \left\|\partial_t u^{\text{DIFF}}(\cdot,t)\right\|_{L^2(\Omega)}$.

For the case $c\equiv 0$ and dimension $d=3$, Ramm [29] establishes an algebraic pointwise convergence and shows that the convergence rate is directly related to the localisation of $q$ and $f$.

Eidus' paper [12] provides an extensive overview of the results available at the time and treats the problem in great generality. In particular, it deals with the wave equation arising from a positive second-order differential operator in divergence form $-\sum_{k,j=1}^{d}\partial_{x_k}\left(a_{kj}(\mathbf{x})\,\partial_{x_j}\right)+q(\mathbf{x})$. It is assumed that $q$ is real-valued and locally Hölder continuous, each $a_{kj}\in C^2\left(\mathbb{R}^d\right)$ is real-valued, $a_{kj}=a_{jk}$, $1\le k,j\le d$, and for any vector $\mathbf{v}\in\mathbb{R}^d$, $\sum_{k,j}^{d} a_{kj}v_kv_j \ge a_0|\mathbf{v}|^2$ with some $a_0>0$; moreover, it is assumed that $|\nabla a_{kj}|$ and $q$ decay fast enough at infinity. The problem is posed in an unbounded domain of $\mathbb{R}^d$ with a finite boundary where the zero Dirichlet boundary condition is imposed. However, it is mentioned in [12, Ch. 2, p. 21] that the obtained results must also hold if this unbounded (exterior) domain is taken to be the whole $\mathbb{R}^d$. The time convergence is proven in $H^1$-norm of the solution and in the $L^2$-norm of its time derivative, with both norms taken over bounded sets.

As a generalization, Vainberg [36], besides geometrical features, also considers higher-order constant coefficient hypoelliptic operators in $\mathbb{R}^d$, whereas Iwasaki [23] treats dissipative wave equations with variable dissipation and potential terms.

Ramm [30] considers a general linear operator and formulates necessary and sufficient conditions for the validity of the LAP in terms of certain properties of the resolvent operator. A more general form of the LAP is formulated, involving time convergence in mean, namely, the convergence of the quantity $\frac{1}{t}\int_0^t e^{i\omega\tau}u(\mathbf{x},\tau)\,d\tau$, for $t\to\infty$, to the stationary solution. This is shown to be equivalent to the validity of the limiting absorption principle.

More recently, the LAP was established for the wave equation of the form (1.2) by Tamura [33], but only in dimension $d=3$ and without quantification of the convergence.

In this brief literature review, we have almost entirely omitted the geometrical issues, which are the most commonly discussed aspects in the literature, see the classical works of Morawetz, e.g. [25], and her collaborators. More on that can also be found in the introductory part of [12]. Finally, we mention some very recent works related to the validity of the LAP for wave propagation in metamaterials [9, 10, 8].

In the present work, we study the LAP for a problem where both material parameters $\alpha$ and $\beta$ are allowed to be nonconstant and prove our results in spatial dimensions $d = 1, 2$ and 3. The main result given in Theorem 1.4 establishes the validity of the LAP and estimates the convergence rates. Additionally, Theorem 1.5 covers the one-dimensional case where a classical formulation of the LAP (i.e. when $U_\infty = 0$) is known not to be valid [11, Sect. 3, Thm. 6]. On a technical side, the novelty of our approach to the proof of the LAP is that it avoids the direct study of the resolvent operator and relies instead on several decay/convergence results. The main features of the present work are:

- The LAP is proven for the wave equation with nonconstant coefficients, which is not necessarily in divergence form (i.e. the equation may have a nonconstant coefficient in front of the divergence operator $\nabla\cdot$). Besides the "classical" case $d = 3$, we also consider $d = 2$.
- The validity of the LAP is extended to the case $d = 1$ with an appropriate modification.
- The convergence in the LAP is quantified and is shown to be algebraic in time for $d = 2, 3$ and exponential for $d = 1$.

We believe that the exponential and algebraic convergence behavior for the cases $d = 1$ and $d = 2$ are generally sharp, but that the rate of the decay for the case $d = 3$ might be improved.

**Outline.** The paper is organised as follows. In Section 2, we state time-decay estimates for the time-domain problem with suitable initial data and source term. In Section 3, we prove the LAP in the form given in Theorems 1.4 and 1.5. The auxiliary time-decay estimates of Section 2 are proven in Section 4. Finally, in Section 5, we summarise the obtained results and give prospects for further work in related directions.

## 2. Time-decay results

In this section, we state some decay-in-time results for solutions to the wave equation with sufficiently localised initial data, which are used in our proof of the LAP in Section 3 below. The proofs of these results are deferred to Section 4. More precisely, we are concerned with the solution of the Cauchy problem

$$\begin{cases} \partial_t^2 u(\mathbf{x},t) - \beta^{-1}(\mathbf{x})\, \nabla\cdot(\alpha(\mathbf{x})\nabla u(\mathbf{x},t)) = f(\mathbf{x},t), & \mathbf{x}\in\mathbb{R}^d,\ t>0,\\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \partial_t u(\mathbf{x},0) = u_1(\mathbf{x}), & \mathbf{x}\in\mathbb{R}^d, \end{cases} \tag{2.1}$$

and its constant-coefficient analog with zero source term:

$$\begin{cases} \partial_t^2 v(\mathbf{x},t) - c_0^2\Delta v(\mathbf{x},t) = 0, \quad \mathbf{x}\in\mathbb{R}^d, \quad t>0,\\ v(\mathbf{x},0) = v_0(\mathbf{x}), \quad \partial_t v(\mathbf{x},0) = v_1(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^d, \end{cases} \tag{2.2}$$

where $c_0 := \sqrt{\alpha_0/\beta_0}$. We start by considering the problem (2.1) in the case of localised initial data and zero source term.

**Proposition 2.1.** *Let $d \geq 2$, $f \equiv 0$. Suppose that $u_0$, $u_1 \in H^2(\mathbb{R}^d)$ and $\alpha$, $\beta$ satisfy Assumptions 1.1 and 1.2. Additionally, the initial data are assumed to satisfy the following localisation condition:*

$$\int_{\mathbb{R}^d} \left(1+|\mathbf{x}|^2\right)^{d+1+\epsilon} \left(|u_0(\mathbf{x})|^2 + |u_1(\mathbf{x})|^2 + |\Delta u_0(\mathbf{x})|^2 + |\Delta u_1(\mathbf{x})|^2\right) d\mathbf{x} < \infty \tag{2.3}$$

*with some $\epsilon > 0$. Denote $\mathbb{R}_+ := [0,\infty)$. Then, for any bounded $\Omega \subset \mathbb{R}^d$, the unique solution $u \in C^2\left(\mathbb{R}_+, L^2\left(\mathbb{R}^d\right)\right) \cap C^1\left(\mathbb{R}_+, H^1\left(\mathbb{R}^d\right)\right) \cap C\left(\mathbb{R}_+, H^2\left(\mathbb{R}^d\right)\right)$ of (2.1) obeys the following decay estimate with some constant $C > 0$, depending on $u_0$, $u_1$, $\alpha$, $\beta$, $\epsilon$, $d$, and $\Omega$:*

$$\|u(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t u(\cdot,t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{\frac{d-1}{2}}}, \quad t \geq 0. \tag{2.4}$$

For the case of zero initial data and a localised source term, we have the following result.

**Proposition 2.2.** *Let $d \geq 2$, $u_0 \equiv 0$, $u_1 \equiv 0$ and $\alpha$, $\beta$ satisfy Assumptions 1.1 and 1.2. Additionally, suppose that $f \in C^1\left(\mathbb{R}_+, L^2\left(\mathbb{R}^d\right)\right) \cap C\left(\mathbb{R}_+, H^1\left(\mathbb{R}^d\right)\right)$, $\bigcup_{t>0} \operatorname{supp} f(\cdot,t) \subset \Omega_f$ for some bounded domain $\Omega_f \subset \mathbb{R}^d$, and there exist constants $C_f$, $p > 0$ such that*

$$\|f(\cdot,t)\|_{L^2(\mathbb{R}^d)} + \|\partial_t f(\cdot,t)\|_{L^2(\mathbb{R}^d)} \leq \frac{C_f}{(1+t^2)^{\frac{p}{2}}}, \quad t \geq 0. \tag{2.5}$$

*Then, for any bounded domain $\Omega \subset \mathbb{R}^d$, the unique solution $u \in C^2\left(\mathbb{R}_+, L^2\left(\mathbb{R}^d\right)\right) \cap C^1\left(\mathbb{R}_+, H^1\left(\mathbb{R}^d\right)\right) \cap C\left(\mathbb{R}_+, H^2\left(\mathbb{R}^d\right)\right)$ of (2.1) obeys the following decay estimates for $t \geq 0$ and some constant $C > 0$ depending on $C_f$, $\alpha$, $\beta$, $p$, $d$, and $\Omega$.*
*For $d = 2$:*

$$\|u(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t u(\cdot,t)\|_{H^1(\Omega)} \leq C \begin{cases} \dfrac{1+\log\left(1+t^2\right)}{(1+t^2)^{\frac{p}{2}}}, & 0 < p \leq 1, \\ \dfrac{1}{(1+t^2)^{\frac{1}{2}}}, & p > 1, \end{cases} \tag{2.6}$$

*For $d > 2$:*

$$\|u(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t u(\cdot,t)\|_{H^1(\Omega)} \leq C \begin{cases} \dfrac{1}{(1+t^2)^{\frac{p}{2}}}, & 0 < p \leq 1, \\ \dfrac{1}{(1+t^2)^{\frac{r}{2}}}, & p > 1, \end{cases} \tag{2.7}$$

*where $r := \min(d-1, p)$.*

Next, we consider the wave equation (2.2) with constant coefficients and $f \equiv 0$.

**Lemma 2.3.** *Let $d = 2, 3$ and $\omega$, $\rho_0 > 0$, $\rho_1 > \rho_0$ be some fixed constants. Let $\mathbb{S}^{d-1} := \left\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\right\}$ be the $(d-1)$-dimensional unit sphere and let $\mathbb{B}_{\rho_0} := \left\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| < \rho_0\right\}$ be the ball of radius $\rho_0$, both centered at $\mathbf{x} = \mathbf{0}$. Fix $\Omega \Subset \mathbb{B}_{\rho_0}$ meaning that $\Omega \subseteq \mathbb{B}_{\rho_0 - \epsilon} \subset \mathbb{B}_{\rho_0}$ for some $\epsilon > 0$. We make the following assumptions on the initial conditions $v_0$ and $v_1$.*

- *For $d = 2$, we assume that*

$$v_0(\mathbf{x}) = A_0(|\mathbf{x}|)\, Y_0\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + V_0(\mathbf{x}), \quad v_1(\mathbf{x}) = A_0(|\mathbf{x}|)\, Y_1\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + V_1(\mathbf{x}), \tag{2.8}$$

*where $A_0 \in C^6(\mathbb{R}_+)$, $Y_0 \in C^6(\mathbb{S}^1)$, $Y_1 \in C^5(\mathbb{S}^1)$, $V_0 \in C_b^6(\mathbb{R}^2)$, $V_1 \in C_b^5(\mathbb{R}^2)$ (we use $C_b$ to denote the space of continuous bounded functions) are such that $A_0(|\mathbf{x}|) = V_0(\mathbf{x}) = V_1(\mathbf{x}) \equiv 0$ for $|\mathbf{x}| \leq \rho_0$, and that there exists a constant $C_0 > 0$ such that*

$$\begin{aligned} &|\mathbf{x}|^{5/2}\left(|V_0(\mathbf{x})| + |V_1(\mathbf{x})| + |\nabla V_0(\mathbf{x})| + |\nabla V_1(\mathbf{x})| + |\Delta V_0(\mathbf{x})|\right. \\ &\left.+ |\Delta V_1(\mathbf{x})| + |\nabla \Delta V_0(\mathbf{x})| + |\nabla \Delta V_1(\mathbf{x})| + \left|\Delta^2 V_0(\mathbf{x})\right| + \left|\Delta^2 V_1(\mathbf{x})\right|\right) \leq C_0 \end{aligned} \tag{2.9}$$

*holds true for all $\mathbf{x} \in \mathbb{R}^2$. Moreover, $A_0(\rho) = e^{i\frac{\omega}{c_0}\rho}/\rho^{3/2}$ for $\rho > \rho_1$.*

- *For $d = 3$, we assume that $v_0 \in C^6(\mathbb{R}^3)$, $v_1 \in C^5(\mathbb{R}^3)$ and that there exists a constant $C_0 > 0$ such that*

$$\begin{aligned} &|\mathbf{x}|^{2}\left(|v_0(\mathbf{x})| + |v_1(\mathbf{x})| + |\nabla v_0(\mathbf{x})| + |\nabla v_1(\mathbf{x})| + |\Delta v_0(\mathbf{x})|\right. \\ &\left.+ |\Delta v_1(\mathbf{x})| + |\nabla \Delta v_0(\mathbf{x})| + |\nabla \Delta v_1(\mathbf{x})| + \left|\Delta^2 v_0(\mathbf{x})\right| + \left|\Delta^2 v_1(\mathbf{x})\right|\right) \leq C_0 \end{aligned} \tag{2.10}$$

*holds true for all $\mathbf{x} \in \mathbb{R}^3$.*

*Then, there exists a constant $C > 0$ such that, for all $\mathbf{x} \in \Omega$ and $t \geq 0$, the unique solution $v \in C^5(\mathbb{R}^d \times \mathbb{R}_+)$ of (2.2) with the initial data as above satisfies*

$$\begin{aligned} &|v(\mathbf{x},t)| + |\nabla v(\mathbf{x},t)| + |\partial_t v(\mathbf{x},t)| \\ &+ |\Delta v(\mathbf{x},t)| + |\partial_t \nabla v(\mathbf{x},t)| + |\partial_t \Delta v(\mathbf{x},t)| \leq \frac{C}{(1+t^2)^{1/2}}. \end{aligned} \tag{2.11}$$

Note that, in the result of Lemma 2.3, we use smoothness and the presence of the oscillatory exponential term in the radial factor in the case $d = 2$ to deduce the $\mathcal{O}(1/t)$ decay instead of the more classical $L^\infty$–decay $\mathcal{O}\left(1/t^{1/2}\right)$ of the solution under absolute integrability and some regularity assumptions on the initial data (see e.g. [4]).

In a similar vein, we can obtain the same decay rate as in Lemma 2.3 for initial data decaying even slower at infinity. To this effect, we require an additional condition, namely, that $v_1$ is the radial derivative of $v_0$ multiplied by $-c_0$.

**Lemma 2.4.** *Let $d = 2, 3$ and $\omega$, $\rho_0 > 0$, $\rho_1 > \rho_0$ be some fixed constants. Using the notation introduced in Lemma 2.3, suppose that $\Omega \Subset \mathbb{B}_{\rho_0}$. Assume that*

$$v_0(\mathbf{x}) = A(|\mathbf{x}|)\, Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), \quad v_1(\mathbf{x}) = -c_0 \partial_{|\mathbf{x}|} v_0(\mathbf{x}) = -c_0 A'(|\mathbf{x}|)\, Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), \tag{2.12}$$

*where $\partial_{|\mathbf{x}|}$ denotes the derivative in the radial direction of the variable $\mathbf{x}$, $A \in C^7(\mathbb{R}_+)$, $Y \in C^7(\mathbb{S}^{d-1})$ such that $A(\rho) \equiv 0$ for $\rho \in [0, \rho_0]$ and $A(\rho) = e^{i\frac{\omega}{c_0}\rho}/\rho^{\frac{d-1}{2}}$ for $\rho > \rho_1$. Then, there exists a constant $C > 0$ such that, for all $\mathbf{x} \in \Omega$ and $t \geq 0$, the unique solution $v \in C^6(\mathbb{R}^d \times \mathbb{R}_+)$ of (2.2) with the initial data (2.12) satisfies*

$$\begin{aligned} &|v(\mathbf{x},t)| + |\nabla v(\mathbf{x},t)| + |\partial_t v(\mathbf{x},t)| \\ &+ |\Delta v(\mathbf{x},t)| + |\partial_t \nabla v(\mathbf{x},t)| + |\partial_t \Delta v(\mathbf{x},t)| \leq \frac{C}{(1+t^2)^{1/2}}. \end{aligned} \tag{2.13}$$

In the one-dimensional case, we have the following exponential decay result, which is proven in [3].

**Proposition 2.5** ([3], Prop. 1.1, Thm. 1.4)**.** *Let $d = 1$ and $f \equiv 0$. Suppose that $u_0 \in H^1(\mathbb{R})$, $u_1 \in L^2(\mathbb{R})$, $\operatorname{supp} u_0$, $\operatorname{supp} u_1 \subset \Omega$ for some bounded $\Omega \subset \mathbb{R}$ and assume $\alpha$, $\beta$, $\Omega_{in}$ be as in Assumption 1.1'. Then, for any bounded $\Omega \subset \mathbb{R}$, the unique solution $u \in C^1\left(\mathbb{R}_+, L^2(\mathbb{R})\right) \cap C\left(\mathbb{R}_+, H^1(\mathbb{R})\right)$ of (2.1) obeys the decay estimate*

$$\|u(\cdot,t) - u_\infty\|_{H^1(\Omega)} + \|\partial_t u(\cdot,t)\|_{L^2(\Omega)} \le C e^{-\Lambda t}, \quad t \ge 0, \tag{2.14}$$

*for some explicit constants $C = C(u_0, u_1, \alpha, \beta, |\Omega|)$, $\Lambda = \Lambda(\alpha,\beta) > 0$ with $|\Omega|$ denoting the Lebesgue measure of the set $\Omega$, and*

$$u_\infty := \frac{1}{2\sqrt{\alpha_0\beta_0}} \int_\Omega u_1(x)\,\beta(x)\,dx. \tag{2.15}$$

## 3. Proof of the LAP (Theorems 1.4 and 1.5)

In this section, we prove Theorems 1.4 and 1.5 at once. Without loss of generality, we can assume that $\Omega = \Omega_{in}$, since both domains could be enlarged to their union without changing the problem. We also suppose that the origin $\mathbf{x} = \mathbf{0}$ is chosen to be inside $\Omega$.

The proof is given in two steps. In Step 1, see Section 3.1, we transform problem (1.2) into an initial-value problem with zero source term for the difference

$$W(\mathbf{x},t) := u(\mathbf{x},t) - e^{-i\omega t} U(\mathbf{x}), \tag{3.1}$$

where $u(\mathbf{x},t)$ and $U(\mathbf{x})$ solve problems (1.2) and (1.1), respectively. In Section 3.2, we observe that the problem introduced in Step 1 has poorly localised initial data, and we write an integral representation, which will be useful in what follows. In Step 2, see Section 3.3, we decompose the problem from Step 1 into several subproblems. We distinguish the cases $d = 1$ and $d \ge 2$. In the former case, the arguments are more transparent and lead to the quantitative result of Theorem 1.5. The higher-dimensional case is more involved, as some of the subproblems do not have sufficiently localised intitial data and thus require the more specific time-decay results given in Section 2.

### 3.1. Step 1: Transformation into an auxiliary homogeneous problem.

By inspection, we see that $W(\mathbf{x},t)$ defined by (3.1) satisfies

$$\begin{cases} \partial_t^2 W(\mathbf{x},t) - \beta^{-1}(\mathbf{x})\,\nabla\cdot(\alpha(\mathbf{x})\,\nabla W(\mathbf{x},t)) = 0, & \mathbf{x}\in\mathbb{R}^d, \quad t>0, \\ W(\mathbf{x},0) = -U(\mathbf{x}), \quad \partial_t W(\mathbf{x},0) = i\omega U(\mathbf{x}), & \mathbf{x}\in\mathbb{R}^d. \end{cases} \tag{3.2}$$

Completing the proofs of Theorems 1.4 and 1.5 is tantamount to showing that there exists a unique constant $U_\infty \in \mathbb{C}$ explicitly given by (1.4) and constants $\Lambda$, $C > 0$ depending on $F$, $\alpha$, $\beta$, $\omega$ such that
for $d = 1$:

$$\|W(\cdot,t) - U_\infty\|_{H^1(\Omega)} + \|\partial_t W(\cdot,t)\|_{L^2(\Omega)} \le C e^{-\Lambda t}, \quad t \ge 0, \tag{3.3}$$

for $d = 2$:

$$\|W(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t W(\cdot,t)\|_{L^2(\Omega)} \le C\,\frac{1+\log\left(1+t^2\right)}{\left(1+t^2\right)^{1/2}}, \quad t \ge 0, \tag{3.4}$$

for $d = 3$:

$$\|W(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t W(\cdot,t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t^2)^{1/2}}, \quad t \geq 0. \tag{3.5}$$

3.2. **Slow decay of the initial data of problem (3.2).** One immediate difficulty when dealing with (3.2) is that the initial data $W(\cdot,0)$ and $\partial_t W(\cdot,0)$ do not belong to $H^1(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$, respectively. The slow decay of the initial conditions in (3.2) can be seen as follows. Let us rewrite (1.1) as the constant-coefficient problem

$$\begin{aligned} -\Delta U(\mathbf{x}) - \frac{\omega^2}{c_0^2} U(\mathbf{x}) =& \frac{1}{\alpha_0} \left[ \beta(\mathbf{x}) F(\mathbf{x}) + (\beta(\mathbf{x}) - \beta_0)\, \omega^2 U(\mathbf{x}) + \right. \\ & \left. + \nabla \cdot (\alpha(\mathbf{x}) \nabla U(\mathbf{x})) - \alpha_0 \Delta U(\mathbf{x}) \right] \\ =:& F_1(\mathbf{x}), \end{aligned} \tag{3.6}$$

where we recall that $c_0^2 = \alpha_0/\beta_0$. Assumptions 1.1 (or 1.1' if $d = 1$) and 1.3 on the coefficients and on $F$ imply that $F_1(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathbb{R}^d \backslash \bar{\Omega}$. Moreover, since the coefficients $\alpha$ and $\beta$ are smooth (for $d \geq 2$) and bounded away from zero, and $F \in L^2(\mathbb{R}^d)$, standard well-posedness results (see e.g. [15, Sec. 6.3.1]) give $U \in H^2(\Omega)$, and hence $F_1 \in L^1(\Omega)$. Therefore, we can write the integral representation of the solution $U$ in $\mathbb{R}^d \backslash \bar{\Omega}$

$$U(\mathbf{x}) = \int_\Omega K(\mathbf{x} - \mathbf{y})\, F_1(\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^d \backslash \bar{\Omega}. \tag{3.7}$$

Here

$$K(\mathbf{x}) := \frac{i}{4} \left( \frac{\omega}{2\pi c_0} \right)^{\frac{d-2}{2}} \frac{1}{|\mathbf{x}|^{\frac{d-2}{2}}} H^{(1)}_{\frac{d-2}{2}} \left( \frac{\omega}{c_0} |\mathbf{x}| \right) \tag{3.8}$$

is the Green's function for the Helmholtz equation (see e.g. [14]) that satisfies the Sommerfeld radiation condition $\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{\frac{d-1}{2}} \left[ \partial_{|\mathbf{x}|} K(\mathbf{x}) - i\frac{\omega}{c_0} K(\mathbf{x}) \right] = 0$ and $-\Delta K(\mathbf{x}) - \frac{\omega^2}{c_0^2} K(\mathbf{x}) = \delta(\mathbf{x})$, with $\delta$ being the $d$-dimensional Dirac delta function. In (3.8), $H_p^{(1)}$ denotes the Hankel function of the first kind of order $p$. Since $\mathbf{y}$ in (3.7) ranges in a bounded set and $F_1 \in L^1(\Omega)$, we employ Lemma A.1 from Appendix A and deduce that

$$U(\mathbf{x}) = \mathcal{O}\left(1/|\mathbf{x}|^{(d-1)/2}\right), \quad \partial_{|\mathbf{x}|} U(\mathbf{x}) - \frac{i\omega}{c_0} U(\mathbf{x}) = \mathcal{O}\left(1/|\mathbf{x}|^{(d+1)/2}\right), \quad |\mathbf{x}| \gg 1. \tag{3.9}$$

This implies that $U$, and therefore $W(\cdot,0)$ and $\partial_t W(\cdot,0)$, do not belong necessarily to $L^2(\mathbb{R}^d)$. At the same time, this gives a precise decay rate in the Sommerfeld radiation condition when the source term $F_1$ in (3.6) is compactly supported.

3.3. **Step 2: Time-decay by decomposition into subproblems.** In order to deal with the slowly decaying initial data in problem (3.2) discussed in Section 3.2, we perform some auxiliary decompositions using the linearity of the problem and the uniqueness of its solution. As $\mathbf{0} \in \Omega$, we can fix $R$ large enough and $\epsilon > 0$ such that $\Omega$ is contained in the open ball $\mathbb{B}_{R-\epsilon} \subset \mathbb{R}^d$ of radius $R-\epsilon$ and center $\mathbf{x} = \mathbf{0}$. Let $\{\eta_0, \eta_1\}$ be a smooth, radial partition of unity, i.e. $\eta_0 = \eta_0(|\mathbf{x}|)$, $\eta_1 = \eta_1(|\mathbf{x}|) \in C^\infty(\mathbb{R}_+)$, and $\eta_0(|\mathbf{x}|) + \eta_1(|\mathbf{x}|) = 1$ for all $\mathbf{x} \in \mathbb{R}^d$, such that

$$\eta_0(|\mathbf{x}|) = \begin{cases} 0, & |\mathbf{x}| < R - \epsilon, \\ 1, & |\mathbf{x}| > R, \end{cases} \qquad \eta_1(|\mathbf{x}|) = \begin{cases} 1, & |\mathbf{x}| < R - \epsilon, \\ 0, & |\mathbf{x}| > R. \end{cases} \tag{3.10}$$

We proceed separately with the case $d = 1$ and the cases $d = 2, 3$.

- **Case $d = 1$ (Theorem 1.5).**

Note that $H^{(1)}_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{ix}$, and hence (3.8) yields, for $d = 1$, $K(|x|) = \frac{i}{2}\frac{c_0}{\omega} e^{i\frac{\omega}{c_0}|x|}$. In this case, the Green function $K$ does not decay at infinity, but the radiation conditions on $K$, and thus on $U$, are exact, i.e. for $x \notin \Omega$, we have $c_0 \partial_{|x|} U(x) = i\omega U(x)$, where $\partial_{|x|} \equiv (\operatorname{sgn} x)\,\partial_x$. Therefore, we can write

$$W(x,t) = \widetilde{W}_0(x,t) + \widetilde{W}_1(x,t), \tag{3.11}$$

where $\widetilde{W}_0(x,t)$, $\widetilde{W}_1(x,t)$ solve the following initial-value problems, respectively:

$$\begin{cases} \partial_t^2 \widetilde{W}_0(x,t) - \beta^{-1}(x)\,\partial_x\Big(\alpha(x)\,\partial_x \widetilde{W}_0(x,t)\Big) = 0, & x \in \mathbb{R}, \quad t > 0, \\ \widetilde{W}_0(x,0) = -\eta_0(|x|)\,U(x), \quad \partial_t \widetilde{W}_0(x,0) = c_0 \partial_{|x|}\Big(\eta_0(|x|)\,U(x)\Big), & x \in \mathbb{R}, \end{cases} \tag{3.12}$$

$$\begin{cases} \partial_t^2 \widetilde{W}_1(x,t) - \beta^{-1}(x)\,\partial_x\Big(\alpha(x)\,\partial_x \widetilde{W}_1(x,t)\Big) = 0, & x \in \mathbb{R}, \quad t > 0, \\ \widetilde{W}_1(x,0) = -\eta_1(|x|)\,U(x), \quad \partial_t \widetilde{W}_1(x,0) = \Big(c_0 \partial_{|x|}\eta_1(|x|) + i\omega \eta_1(|x|)\Big)U(x), & x \in \mathbb{R}. \end{cases} \tag{3.13}$$

Observe that problem (3.12), whose initial data are supported outside $\mathbb{B}_{R-\epsilon}$, is solved by a linear combination of two reflection-free outgoing waves

$$\begin{aligned} \widetilde{W}_0(x,t) = &- H(x - c_0 t)\,\eta_0(|x - c_0 t|)\,U(x - c_0 t) \\ &- H(-x - c_0 t)\,\eta_0(|x + c_0 t|)\,U(x + c_0 t), \end{aligned} \tag{3.14}$$

where $H$ is the Heaviside step function. Note that the smoothness of the solution is not affected by the discontinuity of the Heaviside function due to the vanishing of $\eta_0$. Because of the support property of $\eta_0$, by inspection of (3.14), we have that

$$\widetilde{W}_0(x,t) = \partial_t \widetilde{W}_0(x,t) \equiv 0, \quad x \in \Omega, \quad t > 0. \tag{3.15}$$

To deal with $\widetilde{W}_1$ in (3.11), we observe that the initial data of (3.13) have compact support. Hence, problem (3.13) is amenable to the application of Proposition 2.5, which yields

$$\left\|\widetilde{W}_1(\cdot,t) - U_\infty\right\|_{H^1(\Omega)} + \left\|\partial_t \widetilde{W}_1(\cdot,t)\right\|_{L^2(\Omega)} \le C e^{-\Lambda t}, \quad t \ge 0, \tag{3.16}$$

$$U_\infty := \frac{1}{2\sqrt{\alpha_0\beta_0}} \int_{-R}^{R} \Big( c_0 \partial_{|x|}\eta_1(|x|) + i\omega\eta_1(|x|) \Big) U(x)\,\beta(x)\,dx \tag{3.17}$$
$$= \frac{i\omega}{2\sqrt{\alpha_0\beta_0}} \int_{-R+\epsilon}^{R-\epsilon} U(x)\,\beta(x)\,dx - \frac{1}{2}\left[U(R-\epsilon) + U(-R+\epsilon)\right]$$
$$= \frac{1}{2i\omega\sqrt{\alpha_0\beta_0}} \int_{-R+\epsilon}^{R-\epsilon} F(x)\,\beta(x)\,dx$$

for some constants $C$, $\Lambda > 0$. Note that in passing from the first to the second line in (3.17), $\eta_1$ disappears upon integration by parts using that $\partial_{|x|}U(x) = i\omega/c_0 U(x)$ and $\beta(x) \equiv \beta_0$ for $x \in [-R, -R+\epsilon] \cup [R-\epsilon, R]$. The passage from the second to the third line of the equality is justified upon integration of (1.1) in $x$ over the interval $(-R+\epsilon, R-\epsilon)$ and using again the exact radiation conditions at its endpoints.

Together with (3.15) and (3.11), estimate (3.16) implies (3.3) which completes the proof of Theorem 1.5.

• **Cases $d = 2, 3$ (Theorem 1.4).**

We perform a decomposition, which is similar to (3.11) but contains more terms that have to be treated individually in a more delicate fashion.

Decomposition of $W$. We write the unique solution of (3.2) as

$$W(\mathbf{x}, t) = \sum_{k=1}^{4} W_k(\mathbf{x}, t), \tag{3.18}$$

where $W_1$ solves the homogeneous wave equation

$$\partial_t^2 W_1(\mathbf{x}, t) - \beta^{-1}(\mathbf{x})\,\nabla\cdot\left(\alpha(\mathbf{x})\,\nabla W_1(\mathbf{x}, t)\right) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0,$$

subject to the initial conditions on $\mathbb{R}^d$:

$$W_1(\mathbf{x}, 0) = -\eta_1(|\mathbf{x}|)\,U(\mathbf{x}), \quad \partial_t W_1(\mathbf{x}, 0) = i\omega\eta_1(|\mathbf{x}|)\,U(\mathbf{x}) - c_0\eta_0'(|\mathbf{x}|)\,U_0(\mathbf{x}).$$

$W_2$ and $W_3$ solve the constant-coefficient problems (3.24) and (3.27), respectively, and $W_4$ solves the inhomogeneous wave equation (3.30) with non-constant coefficients. For the initial conditions we use the partition of unity (3.10). The nonzero right-hand side in (3.30) is needed to compensate the fact that the equations in problems (3.24) and (3.27) are different from that in problem (3.2).

Here we have introduced $U_0$, the leading term in the long-range asymptotic expansion of (3.7). More precisely, according to representation (3.7) and Lemma A.1, we have

$$U_0(\mathbf{x}) = \frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{4\pi\,|\mathbf{x}|^{\frac{d-1}{2}}} \left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}} \int_\Omega e^{-i\frac{\omega}{c_0}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}} F_1(\mathbf{y})\,d\mathbf{y}. \tag{3.19}$$

Furthermore, for $|\mathbf{x}| \gg 1$,

(3.20)
$$U(\mathbf{x}) - U_0(\mathbf{x}) = \frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{4\pi |\mathbf{x}|^{\frac{d+1}{2}}} \left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}} \int_\Omega e^{-i\frac{\omega}{c_0}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}} \left[(d-3)(d-1)\frac{ic_0}{8\omega} + \frac{d-1}{2}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} + \frac{i\omega}{2c_0}\left(|\mathbf{y}|^2 - \left(\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)^2\right)\right] F_1(\mathbf{y})\, d\mathbf{y} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right),$$

$$\partial_{|\mathbf{x}|}\left[U(\mathbf{x}) - U_0(\mathbf{x})\right] = -\frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{4\pi |\mathbf{x}|^{\frac{d+1}{2}}} \left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}} \int_\Omega e^{-i\frac{\omega}{c_0}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}} \left[\frac{(d-3)(d-1)}{8} - \frac{d-1}{2}\frac{i\omega}{c_0}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} + \frac{\omega^2}{2c_0^2}\left(|\mathbf{y}|^2 - \left(\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)^2\right)\right] F_1(\mathbf{y})\, d\mathbf{y} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right), \tag{3.21}$$

(3.22)
$$\partial_{|\mathbf{x}|} U(\mathbf{x}) - \frac{i\omega}{c_0} U(\mathbf{x}) = \frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{4\pi |\mathbf{x}|^{\frac{d+1}{2}}} \left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}} \frac{1-d}{2} \int_\Omega e^{-i\frac{\omega}{c_0}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}} F_1(\mathbf{y})\, d\mathbf{y} + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right).$$

Decay of $W_1$. In order to apply Proposition 2.1, we need to check the regularity of the initial conditions of $W_1$. Since $U \in H^2(\Omega)$, we find that $W_1(.,0)$ and the first term of $\partial_t W_1(.,0)$ are in $H^2(\mathbb{R}^d)$, by recalling that $\operatorname{supp}\eta_1 \subset \bar{\mathbb{B}}_R$. The second term of $\partial_t W_1(.,0)$ is in $C^\infty(\mathbb{R}^d)$, as the integral in (3.19) is the Fourier transform (more precisely, its restriction to the unit sphere) of a compactly supported function; see the text below definition (3.6). Moreover, this integral is constant in the radial direction. In addition to being smooth, the second term of $\partial_t W_1(.,0)$ has compact support since $\operatorname{supp}[\eta_0'(|\mathbf{x}|)] \subset \bar{\mathbb{B}}_R \setminus \mathbb{B}_{R-\epsilon}$. We thus conclude that $\partial_t W_1(.,0)$ is also $H^2(\mathbb{R}^d)$.

Since all initial data of $W_1$ are compactly supported, the growth estimate (2.3) clearly holds. Hence, Proposition 2.1 applies to give

$$\|W_1(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t W_1(\cdot,t)\|_{L^2(\Omega)} \le \frac{C}{(1+t^2)^{\frac{d-1}{2}}}, \quad t \ge 0, \tag{3.23}$$

for some constant $C > 0$ depending on $\Omega$.

Decay of $W_2$. $W_2(\mathbf{x},t)$ is the unique solution to the constant-coefficient problem

$$\begin{cases} \partial_t^2 W_2(\mathbf{x},t) - c_0^2 \Delta W_2(\mathbf{x},t) = 0, & \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ W_2(\mathbf{x},0) = \eta_0(|\mathbf{x}|)\left(U_0(\mathbf{x}) - U(\mathbf{x})\right), & \mathbf{x} \in \mathbb{R}^d, \\ \partial_t W_2(\mathbf{x},0) = c_0 \eta_0(|\mathbf{x}|)\left(\frac{i\omega}{c_0} U(\mathbf{x}) - \partial_{|\mathbf{x}|} U_0(\mathbf{x})\right), & \mathbf{x} \in \mathbb{R}^d. \end{cases} \tag{3.24}$$

Note that even though $U \in H^2(\Omega)$, it follows from the smoothness of the kernel function in (3.7) that $U$ is arbitrarily smooth in $\mathbb{R}^d \backslash \bar{\Omega}$. Hence, recalling that $\eta_0$ is zero in $\Omega$, and that $U_0$ is also smooth away from $\mathbf{x}=0$, we deduce that $W_2(\cdot,0),\ \partial_t W_2(\cdot,0)\in C^\infty(\mathbb{R}^d)$. Moreover, since

$$\frac{i\omega}{c_0}U(\mathbf{x})-\partial_{|\mathbf{x}|}U_0(\mathbf{x})=\partial_{|\mathbf{x}|}\left[U(\mathbf{x})-U_0(\mathbf{x})\right]-\left(\partial_{|\mathbf{x}|}U(\mathbf{x})-\frac{i\omega}{c_0}U(\mathbf{x})\right),$$

we see from (3.20)–(3.22) that the initial conditions of (3.24) satisfy the assumptions of Lemma 2.3 with

$$A_0(|\mathbf{x}|):=\eta_0(|\mathbf{x}|)\frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{|\mathbf{x}|^{\frac{d+1}{2}}},$$

$$\begin{aligned}Y_0\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right):=&-\frac{1}{4\pi}\left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}}\int_\Omega e^{-i\frac{\omega}{c_0}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}}\left[(d-3)(d-1)\frac{ic_0}{8\omega}\right.\\&\left.+\frac{d-1}{2}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}+\frac{i\omega}{2c_0}\left(|\mathbf{y}|^2-\left(\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)^2\right)\right]F_1(\mathbf{y})\,d\mathbf{y},\end{aligned}$$

$$\begin{aligned}Y_1\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right):=&-\frac{c_0}{4\pi}\left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}}\int_\Omega e^{-i\frac{\omega}{c_0}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}}\left[\frac{(d-7)(d-1)}{8}\right.\\&\left.-\frac{d-1}{2}\frac{i\omega}{c_0}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}+\frac{\omega^2}{2c_0^2}\left(|\mathbf{y}|^2-\left(\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)^2\right)\right]F_1(\mathbf{y})\,d\mathbf{y},\end{aligned}$$

and $V_0,\ V_1\in C_b^5(\mathbb{R}^d)$ for $d=2,\ 3$. Moreover, the assumptions (2.9) and (2.10) on the initial conditions can be verified using (3.20)–(3.22). Thus Lemma 2.3 entails that the solution $W_2\in C^5(\mathbb{R}^d\times\mathbb{R}_+)$ obeys the following decay estimates uniformly in $\mathbf{x}\in\Omega$ for $t\ge 0$:

$$\begin{aligned}&|W_2(\mathbf{x},t)|+|\nabla W_2(\mathbf{x},t)|+|\partial_t W_2(\mathbf{x},t)|\\&+|\Delta W_2(\mathbf{x},t)|+|\partial_t\nabla W_2(\mathbf{x},t)|+|\partial_t\Delta W_2(\mathbf{x},t)|\le\frac{C}{(1+t^2)^{1/2}}\end{aligned}\tag{3.25}$$

with some constant $C>0$. In particular, (3.25) implies

$$\|W_2(\cdot,t)\|_{H^1(\Omega)}+\|\partial_t W_2(\cdot,t)\|_{L^2(\Omega)}\le\frac{C}{(1+t^2)^{1/2}},\quad t\ge 0.\tag{3.26}$$

Decay of $W_3$. $W_3(\mathbf{x},t)$ is the unique solution to the constant-coefficient problem

$$\begin{cases}\partial_t^2 W_3(\mathbf{x},t)-c_0^2\Delta W_3(\mathbf{x},t)=0, & \mathbf{x}\in\mathbb{R}^d,\quad t>0,\\ W_3(\mathbf{x},0)=-\eta_0(|\mathbf{x}|)U_0(\mathbf{x}),\quad \partial_t W_3(\mathbf{x},0)=c_0\partial_{|\mathbf{x}|}\big(\eta_0(|\mathbf{x}|)U_0(\mathbf{x})\big), & \mathbf{x}\in\mathbb{R}^d.\end{cases}\tag{3.27}$$

As for $W_2$, the initial conditions satisfy $W_3(\cdot,0),\ \partial_t W_3(\cdot,0)\in C^\infty(\mathbb{R}^d)$. Moreover, by setting

$$A(|\mathbf{x}|):=\eta_0(|\mathbf{x}|)\frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{|\mathbf{x}|^{\frac{d-1}{2}}},\qquad Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right):=-\frac{1}{4\pi}\left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}}\int_\Omega e^{-i\frac{\omega}{c_0}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}}F_1(\mathbf{y})\,d\mathbf{y},$$

it is easy to see that (3.27) satisfies the assumptions of Lemma 2.4. Therefore, the solution $W_3 \in C^6\left(\mathbb{R}^d \times \mathbb{R}_+\right)$ obeys the following decay estimate, valid uniformly in $\mathbf{x} \in \Omega$ for $t \geq 0$:

$$\begin{aligned} &|W_3(\mathbf{x},t)| + |\nabla W_3(\mathbf{x},t)| + |\partial_t W_3(\mathbf{x},t)| \\ &+ |\Delta W_3(\mathbf{x},t)| + |\partial_t \nabla W_3(\mathbf{x},t)| + |\partial_t \Delta W_3(\mathbf{x},t)| \leq \frac{C}{\left(1+t^2\right)^{1/2}}, \end{aligned} \tag{3.28}$$

with some constant $C > 0$. In particular,

$$\|W_3(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t W_3(\cdot,t)\|_{L^2(\Omega)} \leq \frac{C}{\left(1+t^2\right)^{1/2}}, \quad t \geq 0. \tag{3.29}$$

Decay of $W_4$. $W_4$ solves the inhomogeneous wave problem

$$\begin{cases} \partial_t^2 W_4(\mathbf{x},t) - \beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla W_4(\mathbf{x},t)) = F_2(\mathbf{x},t) + F_3(\mathbf{x},t), & \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ W_4(\mathbf{x},0) = 0, \quad \partial_t W_4(\mathbf{x},0) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \end{cases} \tag{3.30}$$

where

$$F_k(\mathbf{x},t) := \beta^{-1}(\mathbf{x}) \nabla \alpha(\mathbf{x}) \cdot \nabla W_k(\mathbf{x},t) + \left(\beta^{-1}(\mathbf{x}) \alpha(\mathbf{x}) - c_0^2\right) \Delta W_k(\mathbf{x},t), \; k = 2, 3. \tag{3.31}$$

As already pointed out, the nonzero right-hand side compensates the fact that the equations in problems (3.24) and (3.27) are different from the equation in problem (3.2).

Estimates (3.25) and (3.28) entail the decay of all the terms entering (3.31) and of their time derivative. Hence, recalling the regularity of $W_2$ and $W_3$, we see that Proposition 2.2 is applicable with $p = 1$. This gives the unique solution $W_4 \in C^2\left(\mathbb{R}_+, L^2(\Omega)\right) \cap C^1\left(\mathbb{R}_+, H^1(\Omega)\right) \cap C\left(\mathbb{R}_+, H^2(\Omega)\right)$ which satisties
for $d = 2$:

$$\|W_4(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t W_4(\cdot,t)\|_{L^2(\Omega)} \leq C \frac{1 + \log\left(1+t^2\right)}{\left(1+t^2\right)^{1/2}}, \quad t \geq 0, \tag{3.32}$$

for $d = 3$:

$$\|W_4(\cdot,t)\|_{H^1(\Omega)} + \|\partial_t W_4(\cdot,t)\|_{L^2(\Omega)} \leq \frac{C}{\left(1+t^2\right)^{1/2}}, \quad t \geq 0, \tag{3.33}$$

with some constant $C > 0$.

Consequently, by combining (3.23), (3.26), (3.29), (3.32), and (3.33) with (3.18), the estimates (3.4) and (3.5) follow. This concludes the proof of Theorem 1.4.

## 4. Proofs of the auxiliary time decay results

4.1. **Proof of Proposition 2.1.** This proof is based on an application and an extension of a result from [5]. We shall focus here only on the decay of the solution of (2.1) with $f \equiv 0$. The existence, uniqueness and regularity results are standard. Indeed, whenever $u_0 \in H^{s+1}\left(\mathbb{R}^d\right)$ and $u_1 \in H^s\left(\mathbb{R}^d\right)$ for $s \geq 1$, a direct application of the result from [**?**, Ch. 6, Thm. 4.9] with $f \equiv 0$, together with a bootstrap argument for $\partial_t^2 u$, implies that $u \in C^2\left(\mathbb{R}_+, H^{s-1}\left(\mathbb{R}^d\right)\right) \cap C^1\left(\mathbb{R}_+, H^s\left(\mathbb{R}^d\right)\right) \cap C\left(\mathbb{R}_+, H^{s+1}\left(\mathbb{R}^d\right)\right)$ and

$$\|u(\cdot,t)\|^2_{H^{s+1}(\mathbb{R}^d)}+\|\partial_t u(\cdot,t)\|^2_{H^s(\mathbb{R}^d)}+\left\|\partial_t^2 u(\cdot,t)\right\|^2_{H^{s-1}(\mathbb{R}^d)} \le C\left(\|u_0\|^2_{H^{s+1}(\mathbb{R}^d)}+\|u_1\|^2_{H^s(\mathbb{R}^d)}\right) \tag{4.1}$$

for any $t > 0$ and some constant $C > 0$ that is uniform on any time interval $[0, T]$, $T > 0$. In the present case, since $u_0$, $u_1 \in H^2\left(\mathbb{R}^d\right)$, we have (4.1) with $s = 1$.

Because of Assumption 1.1 on $\alpha$ and $\beta$ (positivity and regularity), the operator $P := -\beta^{-1}(\mathbf{x}) \nabla \cdot (\alpha(\mathbf{x}) \nabla)$ with the domain $\operatorname{Dom} P = H^2\left(\mathbb{R}^d\right)$ is self-adjoint in $L^2_\beta\left(\mathbb{R}^d\right)$ (the $L^2\left(\mathbb{R}^d\right)$ space endowed with the $\beta$-weighted $L^2$ inner product). Note that the sets $L^2_\beta(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ coincide since the weight $\beta$ is bounded and uniformly bounded away from zero. Moreover, $P$ is positive so that there exists a unique self-adjoint, positive operator $B$ with $\operatorname{Dom} B = H^1\left(\mathbb{R}^d\right)$ such that $B^2 = P$. We refer to, e.g. [3, Proof of Prop. 1.1] for a more detailed discussion for the case $d = 1$. With the notation $\sqrt{P} := B$ and $1/\sqrt{P} := B^{-1}$, we can formally write the solution of (2.1) with $f \equiv 0$ as

$$u(\mathbf{x},t) = \cos\left(t\sqrt{P}\right) u_0(\mathbf{x}) + \frac{\sin\left(t\sqrt{P}\right)}{\sqrt{P}} u_1(\mathbf{x}), \quad t \ge 0. \tag{4.2}$$

Under Assumptions 1.1 and 1.2 on $\alpha$ and $\beta$ (compactly supported derivatives and non-trapping), the following operator-norm estimates are obtained in [5, Thm. 1.5]. Namely, there exists a constant $C > 0$ such that

$$\left\| q_\nu^{-1} \frac{\sin\left(t\sqrt{P}\right)}{\sqrt{P}} q_\nu^{-1} \right\|_{L^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)} \le \frac{C}{(1+t^2)^{\frac{d-1}{2}}}, \quad t \ge 0, \tag{4.3}$$

$$\left\| q_\nu^{-1} \cos\left(t\sqrt{P}\right) q_\nu^{-1} \right\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \le \frac{C}{(1+t^2)^{\frac{d}{2}}}, \quad t \ge 0, \tag{4.4}$$

where $q_\nu := \left(1 + |\mathbf{x}|^2\right)^{\nu/2}$ with some $\nu > d + 1$.

Set $\mu := d + 1 + \epsilon$. According to (2.3), we have $q_\mu u_0$ and $q_\mu u_1 \in L^2\left(\mathbb{R}^d\right)$. Then, we deduce from (4.2)–(4.4) that, for $t \ge 0$,

$$\begin{aligned} \left\| q_\mu^{-1} u(\cdot,t) \right\|_{L^2(\mathbb{R}^d)} &\le C\left( \frac{1}{(1+t^2)^{\frac{d}{2}}} \|q_\mu u_0\|_{L^2(\mathbb{R}^d)} + \frac{1}{(1+t^2)^{\frac{d-1}{2}}} \|q_\mu u_1\|_{L^2(\mathbb{R}^d)} \right) \\ &\le \frac{C_0}{(1+t^2)^{\frac{d-1}{2}}} \end{aligned} \tag{4.5}$$

for some constant $C_0 > 0$.

To obtain the estimate for the time derivative $\partial_t u$, we note that $w := \partial_t u$ solves $\partial_t^2 w + P w = 0$, $w(\mathbf{x}, 0) = u_1(\mathbf{x})$, $\partial_t w(\mathbf{x}, 0) = -P u_0(\mathbf{x})$. Hence, we have

$$w(\cdot,t) = \cos\left(t\sqrt{P}\right) u_1 - \frac{\sin\left(t\sqrt{P}\right)}{\sqrt{P}} (P u_0).$$

Therefore, using (4.3) and (4.4), we estimate, for $t \geq 0$,

$$\left\|q_\mu^{-1}\partial_t u\left(\cdot,t\right)\right\|_{L^2\left(\mathbb{R}^d\right)} \leq C\left(\frac{1}{\left(1+t^2\right)^{\frac{d}{2}}}\left\|q_\mu u_1\right\|_{L^2\left(\mathbb{R}^d\right)}+\frac{1}{\left(1+t^2\right)^{\frac{d-1}{2}}}\left\|q_\mu P u_0\right\|_{L^2\left(\mathbb{R}^d\right)}\right) \leq \frac{C_1}{\left(1+t^2\right)^{\frac{d-1}{2}}} \tag{4.6}$$

for some constant $C_1 > 0$.

To complete the $H^1$-estimate of $u$, we estimate the $L^2$-norm of $\nabla u$. First, we observe that $\widetilde{w} := \partial_t^2 u$ solves $\partial_t^2 \widetilde{w} + P\widetilde{w} = 0$, $\widetilde{w}\left(\mathbf{x},0\right) = -Pu_0\left(\mathbf{x}\right)$, $\partial_t \widetilde{w}\left(\mathbf{x},0\right) = -Pu_1\left(\mathbf{x}\right)$. Hence, as before, we have, for $t \geq 0$,

$$\widetilde{w}\left(\cdot,t\right) = -\cos\left(t\sqrt{P}\right)\left(Pu_0\right) - \frac{\sin\left(t\sqrt{P}\right)}{\sqrt{P}}\left(Pu_1\right),$$

$$\left\|q_\mu^{-1}\partial_t^2 u\left(\cdot,t\right)\right\|_{L^2\left(\mathbb{R}^d\right)} \leq C\left(\frac{1}{\left(1+t^2\right)^{\frac{d}{2}}}\left\|q_\mu P u_0\right\|_{L^2\left(\mathbb{R}^d\right)}+\frac{1}{\left(1+t^2\right)^{\frac{d-1}{2}}}\left\|q_\mu P u_1\right\|_{L^2\left(\mathbb{R}^d\right)}\right).$$

We thus arrive at

$$\left\|q_\mu^{-1} P u\left(\cdot,t\right)\right\|_{L^2\left(\mathbb{R}^d\right)} = \left\|q_\mu^{-1}\partial_t^2 u\left(\cdot,t\right)\right\|_{L^2\left(\mathbb{R}^d\right)} \leq \frac{C_2}{\left(1+t^2\right)^{\frac{d-1}{2}}} \tag{4.7}$$

for some constant $C_2 > 0$. Employing the notation $\overline{\left(\cdot\right)}$ for the complex conjugate, we consider the following inner product on $L^2_\beta\left(\mathbb{R}^d\right)$:

$$\left\langle q_\mu^{-1} P u, q_\mu^{-1} u\right\rangle_{L^2_\beta\left(\mathbb{R}^d\right)} = -\int_{\mathbb{R}^d} \nabla\cdot\left(\alpha\left(\mathbf{x}\right)\nabla u\left(\mathbf{x},t\right)\right)\overline{u\left(\mathbf{x},t\right)} q_\mu^{-2}\left(\mathbf{x}\right) d\mathbf{x}. \tag{4.8}$$

In order to elaborate this expression further, we resort to an approximation argument. Let $t > 0$ be fixed. By density of $\mathcal{C}_0^\infty(\mathbb{R}^d)$ in $H^2(\mathbb{R}^d)$, for any $n \in \mathbb{N}$, there exists $u_n\left(\cdot,t\right)$ such that $\left\|u\left(\cdot,t\right) - u_n\left(\cdot,t\right)\right\|_{H^2(\mathbb{R}^d)} \leq 1/n$. For $u_n\left(\cdot,t\right) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, the divergence theorem gives

$$\int_{\mathbb{R}^d} \nabla\cdot\left(\alpha\left(\mathbf{x}\right)\nabla u_n\left(\mathbf{x},t\right)\right)\overline{u\left(\mathbf{x},t\right)} q_\mu^{-2}\left(\mathbf{x}\right) d\mathbf{x} = -\int_{\mathbb{R}^d} \alpha\left(\mathbf{x}\right)\nabla u_n\left(\mathbf{x},t\right)\cdot\nabla\left(\overline{u\left(\mathbf{x},t\right)} q_\mu^{-2}\left(\mathbf{x}\right)\right) d\mathbf{x}.$$

Therefore, adding and subtracting $u_n\left(\cdot,t\right)$ to $u\left(\cdot,t\right)$ under the divergence sign in the right-hand side of (4.8) gives

$$\begin{aligned}\left\langle q_\mu^{-1} P u, q_\mu^{-1} u\right\rangle_{L^2_\beta\left(\mathbb{R}^d\right)} = &-\int_{\mathbb{R}^d} \nabla\cdot\left(\alpha\left(\mathbf{x}\right)\nabla\left(u\left(\mathbf{x},t\right)-u_n\left(\mathbf{x},t\right)\right)\right)\overline{u\left(\mathbf{x},t\right)} q_\mu^{-2}\left(\mathbf{x}\right) d\mathbf{x}\\ &-\int_{\mathbb{R}^d} \alpha\left(\mathbf{x}\right)\nabla\left(u\left(\mathbf{x},t\right)-u_n\left(\mathbf{x},t\right)\right)\cdot\nabla\left(\overline{u\left(\mathbf{x},t\right)} q_\mu^{-2}\left(\mathbf{x}\right)\right) d\mathbf{x}\\ &+\int_{\mathbb{R}^d} \alpha\left(\mathbf{x}\right)\nabla u\left(\mathbf{x},t\right)\cdot\nabla\left(\overline{u\left(\mathbf{x},t\right)} q_\mu^{-2}\left(\mathbf{x}\right)\right) d\mathbf{x}.\end{aligned}$$

The absolute values of the terms on the first and second lines are estimated by multiples of $\|u(\cdot,t)-u_n(\cdot,t)\|_{H^2(\mathbb{R}^d)}\|u(\cdot,t)\|_{H^1(\mathbb{R}^d)}$. Therefore, they vanish in the limit as $n\to\infty$. Consequently, we obtain

$$\begin{aligned}\left\langle q_\mu^{-1}Pu, q_\mu^{-1}u\right\rangle_{L^2_\beta(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\alpha(\mathbf{x})\left|\nabla u(\mathbf{x},t)\right|^2 q_\mu^{-2}(\mathbf{x})\,d\mathbf{x}\\ &\quad+\int_{\mathbb{R}^d}\alpha(\mathbf{x})\overline{u(\mathbf{x},t)}\nabla u(\mathbf{x},t)\cdot\nabla q_\mu^{-2}(\mathbf{x})\,d\mathbf{x}.\end{aligned}$$

Rearranging the terms and employing

$$\left(\nabla u(\mathbf{x},t)\cdot\nabla q_\mu^{-2}(\mathbf{x})\right) = -2\mu\frac{|\mathbf{x}|}{1+|\mathbf{x}|^2}q_\mu^{-2}(\mathbf{x})\,\partial_{|\mathbf{x}|}u(\mathbf{x},t),$$

we arrive at

$$\begin{aligned}\alpha_{\min}\left\|q_\mu^{-1}\nabla u\right\|^2_{L^2(\mathbb{R}^d)} &\le \int_{\mathbb{R}^d}\alpha(\mathbf{x})\left|\nabla u(\mathbf{x},t)\right|^2 q_\mu^{-2}(\mathbf{x})\,d\mathbf{x}\\ &\le \left|\left\langle q_\mu^{-1}Pu, q_\mu^{-1}u\right\rangle_{L^2_\beta(\mathbb{R}^d)}\right| + \mu\|\alpha\|_{L^\infty(\mathbb{R}^d)}\left|\left\langle q_\mu^{-1}\partial_{|\mathbf{x}|}u, q_\mu^{-1}u\right\rangle_{L^2(\mathbb{R}^d)}\right|.\end{aligned}\tag{4.9}$$

Furthermore, employing the Cauchy-Schwarz inequality, we can estimate

$$\left|\left\langle q_\mu^{-1}Pu, q_\mu^{-1}u\right\rangle_{L^2_\beta(\mathbb{R}^d)}\right| \le \|\beta\|_{L^\infty(\mathbb{R}^d)}\left\|q_\mu^{-1}Pu\right\|_{L^2(\mathbb{R}^d)}\left\|q_\mu^{-1}u\right\|_{L^2(\mathbb{R}^d)},$$

$$\begin{aligned}\left|\left\langle q_\mu^{-1}\partial_{|\mathbf{x}|}u, q_\mu^{-1}u\right\rangle_{L^2(\mathbb{R}^d)}\right| &\le \left\|q_\mu^{-1}\nabla u\right\|_{L^2(\mathbb{R}^d)}\left\|q_\mu^{-1}u\right\|_{L^2(\mathbb{R}^d)}\\ &\le \frac{\alpha_{\min}}{4\mu\|\alpha\|_{L^\infty(\mathbb{R}^d)}}\left\|q_\mu^{-1}\nabla u\right\|^2_{L^2(\mathbb{R}^d)} + \frac{\mu\|\alpha\|_{L^\infty(\mathbb{R}^d)}}{\alpha_{\min}}\left\|q_\mu^{-1}u\right\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Here, on the second line, we used the elementary inequality $|a|\,|b|\le\frac{\delta_0}{2}|a|^2+\frac{1}{2\delta_0}|b|^2$, valid for any $\delta_0>0$. Therefore, estimate (4.9) entails

$$\begin{aligned}\frac{3}{4}\alpha_{\min}\left\|q_\mu^{-1}\nabla u\right\|^2_{L^2(\mathbb{R}^d)} &\le \|\beta\|_{L^\infty(\mathbb{R}^d)}\left\|q_\mu^{-1}Pu\right\|_{L^2(\mathbb{R}^d)}\left\|q_\mu^{-1}u\right\|_{L^2(\mathbb{R}^d)}\\ &\quad+\frac{\mu^2}{\alpha_{\min}}\|\alpha\|^2_{L^\infty(\mathbb{R}^d)}\left\|q_\mu^{-1}u\right\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Recalling (4.5) and (4.7), this leads to

$$\left\|q_\mu^{-1}\nabla u\right\|^2_{L^2(\mathbb{R}^d)} \le \frac{4}{3}\left(\frac{C_0C_2}{\alpha_{\min}}\|\beta\|_{L^\infty(\mathbb{R}^d)}+\frac{\mu^2C_0^2}{\alpha_{\min}^2}\|\alpha\|^2_{L^\infty(\mathbb{R}^d)}\right)\frac{1}{(1+t^2)^{d-1}}.\tag{4.10}$$

Finally, denoting with $\chi_\Omega$ the characteristic function of the bounded set $\Omega$, we have

$$\|u\|_{L^2(\Omega)} = \|u\chi_\Omega\|_{L^2(\mathbb{R}^d)} \le C_{\Omega,\mu}\left\|q_\mu^{-1}u\right\|_{L^2(\mathbb{R}^d)},$$

and similarly for $\nabla u$ and $\partial_t u$. Hence, the estimates (4.5), (4.6), and (4.10) furnish (2.4). □

4.2. **Proof of Proposition 2.2.** The existence, uniqueness, and regularity results are standard. Indeed, since $f \in C\left(\mathbb{R}_+, H^1\left(\mathbb{R}^d\right)\right)$, the result from [?, Ch. 6, Thm. 4.9] implies (directly, and by estimating $\partial_t^2 u$ from the wave equation (2.1)) that $u \in C^2\left(\mathbb{R}_+, L^2\left(\mathbb{R}^d\right)\right) \cap C^1\left(\mathbb{R}_+, H^1\left(\mathbb{R}^d\right)\right) \cap C\left(\mathbb{R}_+, H^2\left(\mathbb{R}^d\right)\right)$. We shall focus here only on the decay of the solution. Without loss of generality, we can take $\Omega = \Omega_f$ (by enlarging both sets if necessary). Let $P$, $\sqrt{P}$, and $1/\sqrt{P}$ be defined as at the beginning of Section 4.1. The following operator-norm estimate was obtained in [5, Thm. 1.5]:

$$\left\| \frac{\sin\left(t\sqrt{P}\right)}{\sqrt{P}} \chi_\Omega \right\|_{L^2\left(\mathbb{R}^d\right) \to H^1(\Omega)} \leq \frac{C_0}{\left(1+t^2\right)^{\frac{d-1}{2}}}, \quad t \geq 0, \tag{4.11}$$

for some $C_0 > 0$, where $\chi_\Omega$ denotes the characteristic function of the set $\Omega$.

According to the Duhamel principle, the solution to (2.1) with $u_0 \equiv 0$, $u_1 \equiv 0$ can be written as

$$u\left(\cdot, t\right) = \int_0^t \frac{\sin\left(\left(t-\tau\right)\sqrt{P}\right)}{\sqrt{P}} f\left(\cdot, \tau\right) d\tau. \tag{4.12}$$

Using a basic Bochner integral estimate in $H^1\left(\Omega\right)$ and (4.11), we obtain, for $t > 0$,

$$\begin{aligned} \left\| u\left(\cdot, t\right) \right\|_{H^1(\Omega)} &\leq \int_0^t \left\| \frac{\sin\left(\left(t-\tau\right)\sqrt{P}\right)}{\sqrt{P}} f\left(\cdot, \tau\right) \right\|_{H^1(\Omega)} d\tau \\ &\leq \int_0^t \frac{C_0}{\left(1+\left(t-\tau\right)^2\right)^{\frac{d-1}{2}}} \left\| f\left(\cdot, \tau\right) \right\|_{L^2(\Omega)} d\tau, \end{aligned} \tag{4.13}$$

where, in the second line, we also took into account the assumption that the support of $f\left(\cdot, \tau\right)$ is contained in $\Omega$ for each $\tau > 0$.

Employing the assumed estimate (2.5) on $f$, namely $\left\| f\left(\cdot, \tau\right) \right\|_{L^2(\Omega)} \leq C_f / \left(1+\tau^2\right)^{p/2}$ for some constants $C_f$, $p > 0$ and all $\tau > 0$, and denoting $C := C_0 C_f$, we proceed to estimate

$$\|u(\cdot,t)\|_{H^1(\Omega)} \leq \int_0^t \frac{C\, d\tilde{\tau}}{\left(1+(t-\tilde{\tau})^2\right)^{\frac{d-1}{2}} (1+\tilde{\tau}^2)^{\frac{p}{2}}} \tag{4.14}$$

$$= \frac{C}{t^{d+p-2}} \left[ \int_0^{1/2} \frac{d\tau}{\left(1/t^2+(1-\tau)^2\right)^{\frac{d-1}{2}} (1/t^2+\tau^2)^{\frac{p}{2}}} + \int_{1/2}^{1} \frac{d\tau}{\left(1/t^2+(1-\tau)^2\right)^{\frac{d-1}{2}} (1/t^2+\tau^2)^{\frac{p}{2}}} \right]$$

$$\leq \frac{2^{d-1}C}{t^{p-1}} \frac{1}{(1+t^2)^{\frac{d-1}{2}}} \int_0^{1/2} \frac{d\tau}{(1/t^2+\tau^2)^{\frac{p}{2}}} + \frac{2^p C}{t^{d-2}} \frac{1}{(1+t^2)^{\frac{p}{2}}} \int_0^{1/2} \frac{d\tau}{(1/t^2+\tau^2)^{\frac{d-1}{2}}}.$$

Here we used the change of variable $\tilde{\tau} \mapsto \tau := \tilde{\tau}/t$ and employed the estimates

$$\frac{1/t^{d+p-2}}{\left(1/t^2+(1-\tau)^2\right)^{\frac{d-1}{2}}} \leq \frac{1/t^{d+p-2}}{(1/t^2+1/4)^{\frac{d-1}{2}}} = \frac{2^{d-1}}{t^{p-1}} \frac{1}{(4+t^2)^{\frac{d-1}{2}}} \leq \frac{2^{d-1}}{t^{p-1}} \frac{1}{(1+t^2)^{\frac{d-1}{2}}}, \quad 0 \leq \tau \leq \frac{1}{2},\ t \geq 0,$$

$$\frac{1/t^{d+p-2}}{(1/t^2+\tau^2)^{\frac{p}{2}}} \leq \frac{1/t^{d+p-2}}{(1/t^2+1/4)^{\frac{p}{2}}} = \frac{2^p}{t^{d-2}} \frac{1}{(4+t^2)^{\frac{p}{2}}} \leq \frac{2^p}{t^{d-2}} \frac{1}{(1+t^2)^{\frac{p}{2}}}, \quad \frac{1}{2} \leq \tau \leq 1,\ t \geq 0,$$

in the integrals over $[0,1/2]$ and $[1/2,1]$, respectively. In the last line of (4.14), we have also made the change of variable $\tau \mapsto 1-\tau$. Using Lemma A.2, we continue estimate (4.14):

$$\|u(\cdot,t)\|_{H^1(\Omega)} \leq \frac{2^q C}{(1+t^2)^{\frac{d-1}{2}}} \begin{cases} C_{1,p} t^{1-p}, & 0<p<1, \\ \log\left(t+\sqrt{1+t^2}\right), & p=1, \\ C_{2,p}, & p>1, \end{cases} + \frac{2^q C}{(1+t^2)^{\frac{p}{2}}} \begin{cases} \log\left(t+\sqrt{1+t^2}\right), & d=2, \\ C_{2,d-1}, & d>2, \end{cases} \tag{4.15}$$

where $q := \max(d-1,p)$, $C_{1,s} := \dfrac{1}{1-s}$, $C_{2,s} := \displaystyle\int_0^\infty \frac{dz}{(1+z^2)^{s/2}}$. We continue by considering separately the cases $d=2$ and $d>2$.

Since $C_{1,p} > 1$, estimate (4.15) for $d = 2$ reads

$$\|u(\cdot,t)\|_{H^1(\Omega)} \le 2^q C \begin{cases} C_{1,p}(1+C_{3,p})\dfrac{1+\log(1+t^2)}{(1+t^2)^{\frac{p}{2}}}, & 0<p<1, \\ 2\dfrac{\log\left(t+\sqrt{1+t^2}\right)}{(1+t^2)^{\frac{1}{2}}}, & p=1, \\ (1+C_{4,p})\max(C_{2,p},1)\dfrac{1}{(1+t^2)^{\frac{1}{2}}}, & p>1, \end{cases}$$
$$\le \widetilde{C}_p \begin{cases} \dfrac{1+\log(1+t^2)}{(1+t^2)^{\frac{p}{2}}}, & 0<p\le 1, \\ \dfrac{1}{(1+t^2)^{\frac{1}{2}}}, & p>1, \end{cases} \tag{4.16}$$

where $C_{3,p} := \sup_{t\ge 0} \dfrac{t^{1-p}}{(1+t^2)^{\frac{1-p}{2}}[1+\log(1+t^2)]}$, $C_{4,p} := \sup_{t\ge 0}\dfrac{\log\left(t+\sqrt{1+t^2}\right)}{(1+t^2)^{\frac{p-1}{2}}}$. In (4.16) we also used the elementary estimate

$$\log(t+\sqrt{1+t^2}) \le \log 2 + \frac{1}{2}\log(1+t^2) < 1+\log(1+t^2), \quad t\ge 0.$$

Similarly, when $d > 2$, we have

$$\|u(\cdot,t)\|_{H^1(\Omega)} \le 2^q C \begin{cases} (1+C_{5,d,p})\max(C_{1,p},C_{2,d-1})\dfrac{1}{(1+t^2)^{\frac{p}{2}}}, & 0<p<1, \\ (1+C_{4,d-1})\max(C_{2,d-1},1)\dfrac{1}{(1+t^2)^{\frac{1}{2}}}, & p=1, \\ 2C_{2,r}\dfrac{1}{(1+t^2)^{\frac{r}{2}}}, & p>1, \end{cases}$$
$$\le \widetilde{C}_{p,d} \begin{cases} \dfrac{1}{(1+t^2)^{\frac{p}{2}}}, & 0<p\le 1, \\ \dfrac{1}{(1+t^2)^{\frac{r}{2}}}, & p>1, \end{cases} \tag{4.17}$$

where $r := \min(d-1,p)$, $C_{5,d,p} := \sup_{t\ge 0}\dfrac{t^{1-p}}{(1+t^2)^{\frac{d-1-p}{2}}}$. This completes the estimate of $\|u(\cdot,t)\|_{H^1(\Omega)}$.

To finish the proof, it remains to obtain the estimate for the time derivative $\partial_t u$. To this effect, we note that $w := \partial_t u$ solves $\partial_t^2 w + Pw = \partial_t f$, $w(\mathbf{x},0) = 0$, $\partial_t w(\mathbf{x},0) = f(\mathbf{x},0)$. Hence, we have

$$\partial_t u(\cdot,t) = w(\cdot,t) = \frac{\sin\left(t\sqrt{P}\right)}{\sqrt{P}} f(\cdot,0) + \int_0^t \frac{\sin\left((t-\tau)\sqrt{P}\right)}{\sqrt{P}}\partial_t f(\cdot,\tau)\,d\tau,$$

and consequently we obtain from (4.11), again with $C = C_0 C_f$,

$$\|\partial_t u(\cdot,t)\|_{H^1(\Omega)} \le \frac{C}{(1+t^2)^{\frac{d-1}{2}}} + \int_0^t \left\| \frac{\sin\left((t-\tau)\sqrt{P}\right)}{\sqrt{P}} \partial_t f(\cdot,\tau) \right\|_{H^1(\Omega)} d\tau. \tag{4.18}$$

Therefore, owing to (2.5), the estimate for $\partial_t u$ can be obtained from the estimates for $u$ given in (4.16) and (4.17) by only adding an extra term, which is the first term on the right-hand side of (4.18). Namely, we have, for $d = 2$,

$$\|\partial_t u(\cdot,t)\|_{H^1(\Omega)} \le \begin{cases} \max\left(C, \widetilde{C}_p\right)\left[\frac{1}{(1+t^2)^{\frac{1}{2}}} + \frac{1+\log(1+t^2)}{(1+t^2)^{\frac{p}{2}}}\right], & 0 < p \le 1, \\ \left(C + \widetilde{C}_p\right)\frac{1}{(1+t^2)^{\frac{1}{2}}}, & p > 1, \end{cases} \tag{4.19}$$

$$\le 2\max\left(C, \widetilde{C}_p\right) \begin{cases} \frac{1+\log\left(1+t^2\right)}{(1+t^2)^{\frac{p}{2}}}, & 0 < p \le 1, \\ \frac{1}{(1+t^2)^{\frac{1}{2}}}, & p > 1. \end{cases}$$

Since the first term of the right-hand side of (4.18) decays at least as fast as the second, we have, for $d > 2$,

$$\|\partial_t u(\cdot,t)\|_{H^1(\Omega)} \le \widehat{C}_{p,d} \begin{cases} \frac{1}{(1+t^2)^{\frac{p}{2}}}, & 0 < p \le 1, \\ \frac{1}{(1+t^2)^{\frac{r}{2}}}, & p > 1. \end{cases} \tag{4.20}$$

Altogether, when $d = 2$, estimates (4.19) and (4.16) imply (2.6). Analogously, for $d > 2$, estimates (4.20) and (4.17) furnish (2.7). □

4.3. **Proof of Lemma 2.3.** Since $v_0 \in C^6(\mathbb{R}^d)$, $v_1 \in C^5(\mathbb{R}^d)$, Theorems 2 and 3 in [15, Par. 2.4.1] applied to $u$ and its derivatives (see also (4.76) and (4.113)) imply that the regularity of the solution $v$ to (2.2) is $v \in C^5\left(\mathbb{R}^d \times \mathbb{R}_+\right)$ and $\partial_t v \in C^4\left(\mathbb{R}^d \times \mathbb{R}_+\right)$.

In the main body of the proof, we shall prove that the bound

$$|v(\mathbf{x},t)| \le \frac{C}{(1+t^2)^{1/2}}, \quad \mathbf{x} \in \Omega, \quad t \ge 0, \tag{4.21}$$

is valid for some constant $C > 0$, assuming that $v_0 \equiv 0$.

This first part of the proof actually holds true for weaker regularity assumptions than made in (2.8)-(2.10). More precisely, for $d = 2$, we only need $A_0 \in C^1(\mathbb{R}_+)$, $Y_1 \in C^1(\mathbb{S}^1)$, $V_1 \in C(\mathbb{R}^2)$ with $|\mathbf{x}|^{5/2}|V_1(\mathbf{x})| \le C$, $\mathbf{x} \in \mathbb{R}^2$. And, for $d = 3$, we only need $v_1 \in C(\mathbb{R}^3)$ with $|\mathbf{x}|^2|v_1(\mathbf{x})| \le C$, $\mathbf{x} \in \mathbb{R}^3$. In (4.61) below, we shall summarize these reduced assumptions by saying that $v_1 \in \mathcal{A}$.

The case $v_0 \not\equiv 0$ and the estimate of the other terms in (2.11) is discussed in the final part of this proof. We consider now separately the cases $d = 2$ and $d = 3$.

- **Case** $d = 2$**.**

The solution of (2.2) with $v_0 \equiv 0$ is given by Poisson's formula [15, Par. 2.4.1 (c)]

$$v(\mathbf{x},t) = \frac{t}{2\pi}\int_0^1 \frac{r}{(1-r^2)^{1/2}}\int_{|\mathbf{s}|=1} v_1(\mathbf{x}+\mathbf{s}rc_0t)\,d\sigma_{\mathbf{s}}dr, \tag{4.22}$$

where $d\sigma_{\mathbf{s}}$ denotes the surface measure of the unit circle $\mathbb{S}^1$. Introducing $\rho := |\mathbf{x}+\mathbf{s}rc_0t|$, $\phi := \frac{\mathbf{x}+\mathbf{s}rc_0t}{|\mathbf{x}+\mathbf{s}rc_0t|}$, and using (2.8), we can write

$$\begin{aligned} v(\mathbf{x},t) =& \frac{t}{2\pi}\int_0^1 \frac{r}{(1-r^2)^{1/2}}\int_{|\mathbf{s}|=1} \left[A_0(\rho)Y_1(\phi) + V_1(\mathbf{x}+\mathbf{s}rc_0t)\right]d\sigma_{\mathbf{s}}dr \\ =:& P(\mathbf{x},t) + Q(\mathbf{x},t). \end{aligned} \tag{4.23}$$

As $A_0(\rho) \equiv 0$ for $\rho \le \rho_0$, then $\phi$ is well-defined whenever it appears in the above integral. In fact, $\rho = |\mathbf{x}+\mathbf{s}rc_0t| > \rho_0$ whenever the coefficient $A_0(\rho)$ in front of $Y_1(\phi)$ in (4.23) is different from zero.

We shall prove that there exists some $t_0 > 0$ such that the bounds

$$|P(\mathbf{x},t)| \le \frac{\widetilde{C}}{t}, \qquad |Q(\mathbf{x},t)| \le \frac{\widetilde{C}_0}{t} \tag{4.24}$$

are valid uniformly in $\mathbf{x} \in \Omega$ with some constants $\widetilde{C}$, $\widetilde{C}_0 > 0$ for any $t \ge t_0$. Since it is evident from (4.22) that the solution $v$ is bounded for any finite $t \ge 0$, (4.23) and the estimates in (4.24) will imply (4.21).

Estimate of $Q$ for $t \ge t_0$:

We have

$$Q(\mathbf{x},t) = \frac{t}{2\pi}\int_{a_1/t}^1 \frac{r}{(1-r^2)^{1/2}}\int_{|\mathbf{s}|=1} V_1(\mathbf{x}+\mathbf{s}rc_0t)\,d\sigma_{\mathbf{s}}dr. \tag{4.25}$$

Since $V_1(\mathbf{x}) \equiv 0$ for $|\mathbf{x}| \le \rho_0$, we reduced here the integration range in the $r$ variable from $(0,1)$ to $(a_1/t, 1)$ with

$$a_1 := \frac{1}{c_0}\inf_{|\mathbf{s}|=1,\,\mathbf{x}\in\Omega}\left[\sqrt{(\mathbf{x}\cdot\mathbf{s})^2 + \rho_0^2 - |\mathbf{x}|^2} - \mathbf{x}\cdot\mathbf{s}\right], \tag{4.26}$$

which is positive, due to $\Omega \Subset \mathbb{B}_{\rho_0}$. To justify this reduction of the integration range, we need to prove the implication $r < a_1/t \;\Rightarrow\; \rho < \rho_0$, from which $V_1 \equiv 0$ follows. The condition $r < a_1/t$ means that for $\mathbf{x} \in \Omega$, $\mathbf{s} \in \mathbb{S}^1$, we have

$$rc_0t + \mathbf{x}\cdot\mathbf{s} < \sqrt{(\mathbf{x}\cdot\mathbf{s})^2 + \rho_0^2 - |\mathbf{x}|^2}.$$

For $rc_0t + \mathbf{x}\cdot\mathbf{s} \ge 0$, this is equivalent to

$$\rho^2 = |\mathbf{x}|^2 + 2\mathbf{x}\cdot\mathbf{s}rc_0t + (rc_0t)^2 < \rho_0^2,$$

and $rc_0t + \mathbf{x}\cdot\mathbf{s} < 0$ yields directly

$$\rho^2 = |\mathbf{x}|^2 + 2\mathbf{x}\cdot\mathbf{s}rc_0t + (rc_0t)^2 \le |\mathbf{x}|^2 - (rc_0t)^2 \le |\mathbf{x}|^2 < \rho_0^2.$$

Here and in the sequel we assume that $a_1/t \le 1$, i.e. that $t_0 \ge a_1$. By rearranging the factors, we can write

$$Q(\mathbf{x},t) = \frac{1}{2\pi}\int_{a_1/t}^{1} \frac{(rt)^{-3/2}}{(1-r^2)^{1/2}} \int_{|\mathbf{s}|=1} \left(\frac{rt}{\rho}\right)^{5/2} \rho^{5/2} V_1(\mathbf{x}+\mathbf{s}rc_0t)\, d\sigma_{\mathbf{s}} dr. \tag{4.27}$$

We can assume $\rho = |\mathbf{x}+\mathbf{s}rc_0| > \rho_0$ since the integrand vanishes otherwise due to the support of $V_1$. Thus, we can estimate $rt/\rho$ in (4.27) as follows. From the triangle inequality

$$rc_0t = |\mathbf{x}+\mathbf{s}rc_0t-\mathbf{x}| \le \rho + |\mathbf{x}| \tag{4.28}$$

and $\mathbf{x} \in \Omega \Subset \mathbb{B}_{\rho_0}$, $\rho > \rho_0$, we have

$$\frac{rt}{\rho} \le \frac{1}{c_0}\left(1+\frac{|\mathbf{x}|}{\rho}\right) \le \frac{2}{c_0}. \tag{4.29}$$

Moreover, assumption (2.9) implies $\rho^{5/2}|V_1(\mathbf{x}+\mathbf{s}rc_0t)| \le C_0$. Using this and (4.29), (4.27) gives

$$|Q(\mathbf{x},t)| \le \frac{2^{5/2}C_0}{c_0^{5/2}t^{3/2}}\int_{a_1/t}^{1}\frac{dr}{(1-r^2)^{1/2}r^{3/2}}.$$

If we choose $t_0 := 2a_1$, we obtain for $t \ge t_0$:

$$\begin{aligned}|Q(\mathbf{x},t)| \le & \frac{2^{5/2}C_0}{c_0^{5/2}t^{3/2}}\int_{a_1/t}^{1}\frac{dr}{(1-r^2)^{1/2}r^{3/2}} = \frac{2^{5/2}C_0}{c_0^{5/2}t^{3/2}}\left(\int_{a_1/t}^{1/2}\dots+\int_{1/2}^{1}\dots\right)\\ \le & \frac{2^{7/2}C_0}{\sqrt{3}c_0^{5/2}t^{3/2}}\int_{a_1/t}^{1/2}\frac{dr}{r^{3/2}} + \frac{2^{5/2}C_0}{c_0^{5/2}t^{3/2}}\int_{1/2}^{1}\frac{dr}{(1-r^2)^{1/2}r^{3/2}} \le \frac{\widetilde{C}_0}{t}\end{aligned} \tag{4.30}$$

for some constant $\widetilde{C}_0 > 0$. This completes the proof of the estimate of $Q$ in (4.24) with $t_0 = 2a_1$.

Estimate of $P$ for $t \ge t_0$:

In order to prove the estimate of $P$ in (4.24), let us write

$$\begin{aligned}P(\mathbf{x},t) =& \frac{t}{2\pi}\int_0^1 \frac{e^{ir\omega t}}{(1-r)^{1/2}}\int_{|\mathbf{s}|=1}\left[\frac{re^{-ir\omega t}}{(1+r)^{1/2}}A_0(\rho)Y_1(\phi)\right.\\ &\left.-\left(\frac{e^{-i\omega t}}{\sqrt{2}}-\frac{e^{-i\omega t}}{\sqrt{2}}\right)A_0(|\mathbf{x}+\mathbf{s}c_0t|)Y_1\left(\frac{\mathbf{x}+\mathbf{s}c_0t}{|\mathbf{x}+\mathbf{s}c_0t|}\right)\right]d\sigma_{\mathbf{s}}dr\\ =& \frac{t}{2\pi}\int_0^1 \frac{e^{ir\omega t}}{(1-r)^{1/2}}\int_{|\mathbf{s}|=1}\left[\frac{re^{-ir\omega t}}{(1+r)^{1/2}}A_0(\rho)Y_1(\phi)\right.\\ &\left.-\frac{e^{-i\omega t}}{\sqrt{2}}A_0(|\mathbf{x}+\mathbf{s}c_0t|)Y_1\left(\frac{\mathbf{x}+\mathbf{s}c_0t}{|\mathbf{x}+\mathbf{s}c_0t|}\right)\right]d\sigma_{\mathbf{s}}dr\\ &+\frac{t}{2\sqrt{2}\pi}\int_{|\mathbf{s}|=1}A_0(|\mathbf{x}+\mathbf{s}c_0t|)Y_1\left(\frac{\mathbf{x}+\mathbf{s}c_0t}{|\mathbf{x}+\mathbf{s}c_0t|}\right)d\sigma_{\mathbf{s}}\int_0^1\frac{e^{-i(1-r)\omega t}}{(1-r)^{1/2}}dr\\ =:& P_1(\mathbf{x},t)+P_2(\mathbf{x},t).\end{aligned} \tag{4.31}$$

We start with

$$P_2(\mathbf{x},t) = \frac{1}{t^{1/2}} F_2(\mathbf{x},t) \int_0^1 \frac{e^{-ir\omega t}}{r^{1/2}} dr, \tag{4.32}$$

where we made a change of variable $r \mapsto (1-r)$ and introduced

$$F_2(\mathbf{x},t) := \frac{t^{3/2}}{2^{3/2}\pi} \int_{|\mathbf{s}|=1} A_0(|\mathbf{x}+\mathbf{s}c_0 t|)\, Y_1\left(\frac{\mathbf{x}+\mathbf{s}c_0 t}{|\mathbf{x}+\mathbf{s}c_0 t|}\right) d\sigma_{\mathbf{s}}. \tag{4.33}$$

Using (4.28), the assumed form of $A_0$ and (4.29), both for $\rho > \rho_1$, we have uniformly for $\mathbf{x} \in \Omega$, $\mathbf{s} \in \mathbb{S}^1$,

$$\begin{aligned}(rt)^{3/2} |A_0(\rho)| \le& \begin{cases} \left(\frac{\rho+\rho_0}{c_0}\right)^{3/2} \|A_0\|_{L^\infty(\mathbb{R}_+)}, & 0 \le \rho \le \rho_1, \\ \left(\frac{rt}{\rho}\right)^{3/2} \le \left(\frac{2}{c_0}\right)^{3/2}, & \rho > \rho_1, \end{cases} \\ \le& C_1, \quad \rho \ge 0, \end{aligned} \tag{4.34}$$

for some constant $C_1 > 0$. Thus, using (4.34) with $r = 1$ and recalling the assumptions on $Y_1$, we deduce

$$\sup_{\mathbf{x}\in\Omega} \|F_2(\mathbf{x},\cdot)\|_{L^\infty(\mathbb{R}_+)} =: C_2 < \infty. \tag{4.35}$$

Finally, employing Lemma A.3 from Appendix A, we obtain from (4.32) and (4.35), for $\mathbf{x} \in \Omega$ and $t \ge t_0$,

$$|P_2(\mathbf{x},t)| \le \frac{\widetilde{C}_2}{t} \tag{4.36}$$

with some constant $\widetilde{C}_2 > 0$ and any $t_0 > 0$.

Decay of $P_1$. To deal with $P_1$, we note that the integrand is a smooth function of $r$ in $[0,1)$ and it behaves like $(1-r)^{1/2}$ as $r \to 1$. Integrating by parts in the $r$ variable with $e^{ir\omega t} dr$ as differential, both boundary terms vanish (recall also that $A_0(|\mathbf{x}|) \equiv 0$ for $\mathbf{x} \in \Omega$). We thus arrive at

$$\begin{aligned} P_1(\mathbf{x},t) =& -\frac{1}{2\pi i\omega} \int_0^1 \int_{|\mathbf{s}|=1} e^{ir\omega t} \partial_r \left( \frac{1}{(1-r)^{1/2}} \left[ \frac{re^{-ir\omega t}}{(1+r)^{1/2}} A_0(\rho)\, Y_1(\phi) \right.\right. \\ & \left.\left. -\frac{e^{-i\omega t}}{\sqrt{2}} A_0(|\mathbf{x}+\mathbf{s}c_0 t|)\, Y_1\left(\frac{\mathbf{x}+\mathbf{s}c_0 t}{|\mathbf{x}+\mathbf{s}c_0 t|}\right)\right]\right) d\sigma_{\mathbf{s}} dr \\ =& I_1(\mathbf{x},t) + I_2(\mathbf{x},t) + I_3(\mathbf{x},t) + I_4(\mathbf{x},t), \end{aligned} \tag{4.37}$$

where

$$I_1(\mathbf{x},t) := \frac{i}{2\pi\omega} \int_0^1 \int_{|\mathbf{s}|=1} \frac{r}{(1-r^2)^{1/2}} Y_1(\phi)\, e^{ir\omega t} \partial_r \left(e^{-ir\omega t} A_0(\rho)\right) d\sigma_{\mathbf{s}} dr, \tag{4.38}$$

$$I_2(\mathbf{x},t) := \frac{i}{2\pi\omega} \int_0^1 \int_{|\mathbf{s}|=1} \frac{r}{(1-r^2)^{1/2}} A_0(\rho)\, \partial_r Y_1(\phi)\, d\sigma_{\mathbf{s}} dr, \tag{4.39}$$

$$I_3(\mathbf{x},t) := \frac{i}{4\pi\omega}\int_0^1\int_{|\mathbf{s}|=1}\frac{2+r}{(1-r)^{1/2}(1+r)^{3/2}}A_0(\rho)\,Y_1(\phi)\,d\sigma_{\mathbf{s}}dr, \tag{4.40}$$

$$\begin{aligned}I_4(\mathbf{x},t) :=& \frac{i}{4\pi\omega}\int_0^1\int_{|\mathbf{s}|=1}\frac{1}{(1-r)^{3/2}}\left[\frac{r}{(1+r)^{1/2}}A_0(\rho)\,Y_1(\phi)\right.\\ &\left.-\frac{e^{-i(1-r)\omega t}}{\sqrt{2}}A_0(|\mathbf{x}+\mathbf{s}c_0t|)\,Y_1\left(\frac{\mathbf{x}+\mathbf{s}c_0t}{|\mathbf{x}+\mathbf{s}c_0t|}\right)\right]d\sigma_{\mathbf{s}}dr.\end{aligned} \tag{4.41}$$

Here and in the sequel, we use the following notation to avoid having too many brackets: $\partial_r A_0(\rho) := \partial_r(A_0(\rho))$, $\partial_r Y_1(\phi) := \partial_r(Y_1(\phi))$, $\nabla Y_1(\phi) := (\nabla Y_1)(\phi)$.

For the sake of the proof, we shall extend the function $Y_1$ from the circle $\mathbb{S}^1$ to a neighbourhood of it, e.g. to the annulus with the inner and outer radii $\frac{1}{2}$ and $\frac{3}{2}$, respectively. We choose a constant extension in the radial direction, as this will simplify the proof. In fact, this extension makes $\nabla Y_1$ well defined on $\mathbb{S}^1$. Then, $\nabla Y_1(\phi)$ is a tangent vector to $\mathbb{S}^1$ for each (radial vector) $\phi \in \mathbb{S}^1$, and hence

$$\phi \cdot \nabla Y_1(\phi) = 0. \tag{4.42}$$

Decay of $I_1$. We have

$$\partial_r A_0(\rho) = c_0 t A_0'(\rho)\frac{\mathbf{x}\cdot\mathbf{s}+rc_0t}{|\mathbf{x}+\mathbf{s}rc_0t|}, \quad \rho > 0. \tag{4.43}$$

Moreover, since $|\mathbf{x}+\mathbf{s}rc_0t| = rc_0t\left(1+2\frac{\mathbf{x}\cdot\mathbf{s}}{rc_0t}+\frac{|\mathbf{x}|^2}{r^2c_0^2t^2}\right)^{1/2}$ for $|\mathbf{s}|=1$, the estimate

$$1-\frac{\mathbf{x}\cdot\mathbf{s}+rc_0t}{|\mathbf{x}+\mathbf{s}rc_0t|} = \frac{|\mathbf{x}|^2-(\mathbf{x}\cdot\mathbf{s})^2}{2r^2c_0^2t^2}+\mathcal{O}\left(\frac{1}{r^3t^3}\right) = \mathcal{O}\left(\frac{1}{r^2t^2}\right) \tag{4.44}$$

is valid for $rt \gg 1$. This can be seen from the Taylor expansion of $(1+w)^{-1/2}$ around zero, with $w := 2\frac{\mathbf{x}\cdot\mathbf{s}}{rc_0t}+\frac{|\mathbf{x}|^2}{r^2c_0^2t^2}$.

Then, we can write

$$\begin{aligned}\frac{r^{5/2}t^{3/2}}{c_0}e^{ir\omega t}\partial_r\left(e^{-ir\omega t}A_0(\rho)\right) =& (rt)^{5/2}\left(A_0'(\rho)-\frac{i\omega}{c_0}A_0(\rho)\right)\\ &-(rt)^{5/2}A_0'(\rho)\left(1-\frac{\mathbf{x}\cdot\mathbf{s}+rc_0t}{|\mathbf{x}+\mathbf{s}rc_0t|}\right), \quad \rho>0,\end{aligned}$$

where both terms on the right-hand side are uniformly bounded for $rt > a_1$ (and hence $\rho > \rho_0$), $\mathbf{x}\in\Omega$, $|\mathbf{s}|=1$. This can be deduced from (4.44) using (4.29) and the estimates $\left|A_0'(\rho)-\frac{i\omega}{c_0}A_0(\rho)\right| = 3/(2\rho^{5/2})$, $|A_0'(\rho)| \le C/\rho^{3/2}$ for $\rho>\rho_1$ and some constant $C>0$. Therefore, we have for

$$F_3(\mathbf{x},rt) := \frac{ic_0}{2\pi\omega}\int_{|\mathbf{s}|=1}Y_1(\phi)\frac{r^{5/2}t^{3/2}}{c_0}e^{ir\omega t}\partial_r\left(e^{-ir\omega t}A_0(\rho)\right)d\sigma_{\mathbf{s}} : \tag{4.45}$$

$$\sup_{\mathbf{x}\in\Omega}\|F_3(\mathbf{x},\cdot)\|_{L^\infty(a_1,\infty)} =: C_3 < \infty.$$

Since both $A_0$, $A_0'$ vanish on $[0,\rho_0]$, the integrals in $r$ in each of (4.38)–(4.40) reduces to $(a_1/t,1)$ (see the discussion before and after (4.26)). Hence we can estimate $I_1$ in (4.38) for $t \geq t_0 := 2a_1$ as

$$|I_1(\mathbf{x},t)| = \frac{1}{t^{3/2}}\left|\int_{a_1/t}^{1}\frac{1}{r^{3/2}(1-r^2)^{1/2}}F_3(\mathbf{x},rt)\,dr\right| \leq \frac{2C_3}{3^{1/2}t^{3/2}}\int_{a_1/t}^{1/2}\frac{dr}{r^{3/2}} + \frac{2^{3/2}C_3}{t^{3/2}}\int_{1/2}^{1}\frac{dr}{(1-r)^{1/2}} \leq \frac{\widetilde{C}_3}{t} \tag{4.46}$$

with some constant $\widetilde{C}_3 > 0$. In a similar but simpler fashion we can estimate the terms $I_2$ and $I_3$.

Decay of $I_2$. Since

$$\partial_r Y_1(\phi) = \frac{c_0 t\mathbf{s}\cdot\nabla Y_1(\phi)}{|\mathbf{x}+\mathbf{s}rc_0t|}, \tag{4.47}$$

where we used (4.42), for

$$F_4(\mathbf{x},rt) := \frac{ic_0(rt)^{5/2}}{2\pi\omega}\int_{|\mathbf{s}|=1}A_0(\rho)\frac{\mathbf{s}\cdot\nabla Y_1(\phi)}{|\mathbf{x}+\mathbf{s}rc_0t|}\,d\sigma_{\mathbf{s}}, \tag{4.48}$$

we have

$$|F_4(\mathbf{x},rt)| \leq \frac{c_0}{\pi\omega}\int_{|\mathbf{s}|=1}\frac{rt}{\rho}(rt)^{3/2}|A_0(\rho)|\,|\nabla Y_1(\phi)|\,d\sigma_{\mathbf{s}}.$$

Hence, using (4.29) and (4.34), we deduce

$$\sup_{\mathbf{x}\in\Omega}\|F_4(\mathbf{x},\cdot)\|_{L^\infty(a_1,\infty)} =: C_4 < \infty.$$

Therefore, we obtain for $t \geq t_0 = 2a_1$

$$|I_2(\mathbf{x},t)| = \frac{1}{t^{3/2}}\left|\int_{a_1/t}^{1}\frac{1}{r^{3/2}(1-r^2)^{1/2}}F_4(\mathbf{x},rt)\,dr\right| \leq \frac{2C_4}{3^{1/2}t^{3/2}}\int_{a_1/t}^{1/2}\frac{dr}{r^{3/2}} + \frac{2^{3/2}C_4}{t^{3/2}}\int_{1/2}^{1}\frac{dr}{(1-r)^{1/2}} \leq \frac{\widetilde{C}_4}{t}. \tag{4.49}$$

Decay of $I_3$. Similarly,

$$F_5(\mathbf{x},rt) := \frac{i(rt)^{3/2}}{4\pi\omega}\int_{|\mathbf{s}|=1}A_0(\rho)\,Y_1(\phi)\,d\sigma_{\mathbf{s}}, \tag{4.50}$$

satisfies

$$|F_5(\mathbf{x},rt)| = \frac{1}{4\pi\omega}\int_{|\mathbf{s}|=1}(rt)^{3/2}|A_0(\rho)|\,|Y_1(\phi)|\,d\sigma_{\mathbf{s}}.$$

Hence, using again (4.34),

$$\sup_{\mathbf{x}\in\Omega}\|F_5(\mathbf{x},\cdot)\|_{L^\infty(0,\infty)} =: C_5 < \infty. \tag{4.51}$$

Therefore, we obtain for $t \geq t_0 = 2a_1$

$$|I_3(\mathbf{x},t)| = \frac{1}{t^{3/2}} \left| \int_{a_1/t}^{1} \frac{2+r}{r^{3/2}(1-r)^{1/2}(1+r)^{3/2}} F_5(\mathbf{x},rt)\, dr \right| \tag{4.52}$$
$$\leq \frac{5C_5}{2^{1/2}t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} + \frac{8C_5}{3^{1/2}t^{3/2}} \int_{1/2}^{1} \frac{dr}{(1-r)^{1/2}} \leq \frac{\widetilde{C}_5}{t}$$

with some constants $\widetilde{C}_4$, $\widetilde{C}_5 > 0$.

Decay of $I_4$. To treat the term $I_4$, we introduce

$$\widetilde{F}_6(\mathbf{x},r,t) := \int_{|\mathbf{s}|=1} \left[ \frac{r}{(1+r)^{1/2}} A_0(\rho)\, Y_1(\phi) - \frac{e^{-i(1-r)\omega t}}{\sqrt{2}} A_0(|\mathbf{x}+\mathbf{s}c_0 t|)\, Y_1\left(\frac{\mathbf{x}+\mathbf{s}c_0 t}{|\mathbf{x}+\mathbf{s}c_0 t|}\right) \right] d\sigma_{\mathbf{s}}, \tag{4.53}$$

$$F_6(\mathbf{x},r,t) := \frac{1}{1-r} \widetilde{F}_6(\mathbf{x},r,t). \tag{4.54}$$

Using $\widetilde{F}_6(\mathbf{x},1,t) = 0$, we rewrite (4.54) as

$$F_6(\mathbf{x},r,t) = -\frac{1}{1-r}\left(\widetilde{F}_6(\mathbf{x},1,t) - \widetilde{F}_6(\mathbf{x},r,t)\right) = -\frac{1}{1-r}\int_r^1 \partial_r \widetilde{F}_6(\mathbf{x},\tau,t)\, d\tau. \tag{4.55}$$

Then, for $r \in (1-a_1/t, 1)$, we estimate

$$|F_6(\mathbf{x},r,t)| \leq \left\| \partial_r \widetilde{F}_6(\mathbf{x},\cdot,t) \right\|_{L^\infty(1-a_1/t,1)} \tag{4.56}$$
$$\leq \int_{|\mathbf{s}|=1} \Bigg[ \left( \frac{1}{2(1+r)^{3/2}} + \frac{1}{(1+r)^{1/2}} \right) |A_0(\rho)|\, |Y_1(\phi)| + \frac{1}{(1+r)^{1/2}} |\partial_r A_0(\rho)|\, |Y_1(\phi)| + \frac{1}{(1+r)^{1/2}} |A_0(\rho)|\, |\partial_r Y_1(\phi)| + \frac{\omega t}{\sqrt{2}} |A_0(|\mathbf{x}+\mathbf{s}c_0 t|)| \left| Y_1\left(\frac{\mathbf{x}+\mathbf{s}c_0 t}{|\mathbf{x}+\mathbf{s}c_0 t|}\right)\right| \Bigg] d\sigma_{\mathbf{s}}.$$

For the term with $\partial_r A_0$ we first use (4.43) and consider the following estimate which holds uniformly for $\mathbf{x} \in \Omega$, $\mathbf{s} \in \mathbb{S}^1$. It is obtained in analogy to (4.34).

$$(rt)^{3/2} |A_0'(\rho)| \leq \begin{cases} \left(\frac{\rho+\rho_0}{c_0}\right)^{3/2} \|A_0'\|_{L^\infty(\mathbb{R}_+)}, & 0 \leq \rho \leq \rho_1, \\ \left(\frac{rt}{\rho}\right)^{3/2} \left(\frac{\omega}{c_0} + \frac{3}{2\rho_1}\right) \leq \left(\frac{2}{c_0}\right)^{3/2} \left(\frac{\omega}{c_0} + \frac{3}{2\rho_1}\right), & \rho > \rho_1, \end{cases}$$
$$\leq \widetilde{C}_1, \quad \rho \geq 0,$$

for some constant $\widetilde{C}_1 > 0$.

For the term with $\partial_r Y_1$ we employ (4.34), $r \geq 1 - \frac{a_1}{t} \geq \frac{1}{2}$ (for $t \geq t_0 = 2a_1$), and

$$|\partial_r Y_1(\phi)| \leq \frac{2c_0 t |\nabla Y_1(\phi)|}{\rho} \leq \frac{2c_0 t}{\rho_0} \|\nabla Y_1\|_{L^\infty(\mathbb{S}^1)},$$

where we used (4.47) and the fact that $A_0$ vanishes on $[0, \rho_0]$.

Therefore, employing in (4.56) the estimate (4.34) with $r = 1$ we deduce that

$$\sup_{\mathbf{x}\in\Omega,\ r\in(1-a_1/t,1),\ t>2a_1} \left|F_6(\mathbf{x},r,t)\,t^{1/2}\right| =: C_6 < \infty. \tag{4.57}$$

Moreover, for $r \in (1/2, 1 - a_1/t)$ and for $\epsilon \in (0, 1/2]$, taking into account (4.54), we have

$$\left|(1-r)^{1/2+\epsilon}\,t^{1+\epsilon}F_6(\mathbf{x},r,t)\right| = \left|\widetilde{F}_6(\mathbf{x},r,t)\right| \frac{t^{1+\epsilon}}{(1-r)^{1/2-\epsilon}} \le \left|\widetilde{F}_6(\mathbf{x},r,t)\right| \frac{t^{3/2}}{a_1^{1/2-\epsilon}}.$$

Hence, with the constants $C_2$ and $C_5$ introduced in (4.35) and (4.51), respectively, we obtain from (4.53) that, for any $\epsilon \in (0, 1/2]$,

$$\begin{aligned}\sup_{\mathbf{x}\in\Omega,\ r\in(1/2,1-a_1/t),\ t>2a_1} \left|(1-r)^{1/2+\epsilon}\,t^{1+\epsilon}F_6(\mathbf{x},r,t)\right| \le &\frac{1}{a_1^{1/2-\epsilon}}\left(\frac{8\pi\omega}{3^{1/2}}C_5 + 2\pi C_2\right)\\ =:&C_7 < \infty.\end{aligned} \tag{4.58}$$

Bounds (4.58) and (4.57) imply that, for $t \ge t_0 = 2a_1$ and $\epsilon \in (0, 1/2]$, we get

$$\begin{aligned}|I_4(\mathbf{x},t)| \le &\frac{1}{4\pi\omega}\int_0^1 \frac{1}{(1-r)^{1/2}}\,|F_6(\mathbf{x},r,t)|\,dr\\ =&\frac{1}{4\pi\omega}\left(\int_0^{1/2}\dots + \int_{1/2}^{1-a_1/t}\dots + \int_{1-a_1/t}^{1}\dots\right)\\ \le&\frac{2^{3/2}}{t^{3/2}}\left(C_5\int_0^{1/2}\frac{dr}{r^{1/2}} + \frac{C_2}{4\omega}\right) + \frac{C_7}{4\pi\omega t^{1+\epsilon}}\int_{1/2}^{1-a_1/t}\frac{dr}{(1-r)^{1+\epsilon}}\\ &+ \frac{C_6}{4\pi\omega t^{1/2}}\int_{1-a_1/t}^{1}\frac{dr}{(1-r)^{1/2}} \le \frac{\widetilde{C}_6}{t}\end{aligned} \tag{4.59}$$

with some constant $\widetilde{C}_6 > 0$. For the interval $(0, 1/2)$, we estimated here the integrand directly from (4.41), using (4.51) and (4.35).

From estimates (4.46), (4.49), (4.52), and (4.59) of the terms $I_1$, $I_2$, $I_3$, and $I_4$, respectively, in decomposition (4.37), we obtain

$$|P_1(x,t)| \le \frac{C_8}{t}$$

with some constant $C_8 > 0$ and $t_0 = 2a_1$. Together with (4.36) this gives the estimate of $P$ in (4.24), again with $t_0 = 2a_1$. This concludes the proof of (4.21) in the case $d = 2$.

- **Case** $d = 3$.

The solution is given by Kirchhoff's formula [15, Par. 2.4.1 (c)]

$$v(\mathbf{x},t) = \frac{1}{4\pi}\int_{|\mathbf{s}|=1} t\,v_1(\mathbf{x} + \mathbf{s}c_0 t)\,d\sigma_{\mathbf{s}}, \tag{4.60}$$

where $d\sigma_{\mathbf{s}}$ denotes the surface measure of the unit sphere $\mathbb{S}^2$. In this case, it is immediate to see that (4.60) implies (4.21), owing to assumption (2.10).

• **Conclusion of the proof.** So far, we have proved the decay of the solution under the assumption $v_0 \equiv 0$. We now extend the result to the general case and show that estimates analogous to (4.21) hold true for the derivatives of the solution.

To proceed, it is convenient to introduce the following notation. Given a function $w$ and constants $\omega$, $c_0$, $\rho_0 > 0$, $\rho_1 > \rho_0$, we say that

$$w \in \mathcal{A} \equiv \mathcal{A}_{\omega,c_0,\rho_0,\rho_1} \tag{4.61}$$

if the following conditions are satisfied:

- When $d = 2$, we can write

$$w(\mathbf{x}) = A_w(|\mathbf{x}|) Y_w\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + V_w(\mathbf{x})$$

for some functions $A_w \in C^1(\mathbb{R}_+)$, $Y_w \in C^1(\mathbb{S}^1)$, $V_w \in C(\mathbb{R}^2)$ such that

$$A_w(|\mathbf{x}|) \equiv 0 \equiv V_w(\mathbf{x}), \quad |\mathbf{x}| \leq \rho_0, \qquad A_w(\rho) = \frac{e^{i\frac{\omega}{c_0}\rho}}{\rho^{3/2}}, \quad \rho > \rho_1,$$

and we have

$$|\mathbf{x}|^{5/2} |V_w(\mathbf{x})| \leq C, \quad \mathbf{x} \in \mathbb{R}^2,$$

for some constant $C > 0$.

- When $d = 3$, we have $w \in C(\mathbb{R}^3)$ and

$$|\mathbf{x}|^2 |w(\mathbf{x})| \leq C, \quad \mathbf{x} \in \mathbb{R}^3,$$

for some constant $C > 0$.

Let us denote $Z[v_0, v_1] \equiv Z$ the solution of the wave equation $\partial_t^2 Z(\mathbf{x}, t) - c_0^2 \Delta Z(\mathbf{x}, t) = 0$ for $\mathbf{x} \in \mathbb{R}^d$, $t > 0$, subject to the initial conditions $Z(\mathbf{x}, 0) = v_0(\mathbf{x})$, $\partial_t Z(\mathbf{x}, 0) = v_1(\mathbf{x})$. The assumptions $v_0 \in C^6\left(\mathbb{R}^d\right)$ and $v_1 \in C^5\left(\mathbb{R}^d\right)$ entail

$$Z \in C^5\left(\mathbb{R}^d \times \mathbb{R}_+\right), \quad \partial_t Z \in C^4\left(\mathbb{R}^d \times \mathbb{R}_+\right); \tag{4.62}$$

recall Theorems 2 and 3 in [15, Par. 2.4.1] (applied to $Z$ and its derivatives), see also (4.76) and (4.113).

We recall that, in proving the decay for $Z[0, v_1]$ given by (4.21), we have only used that $v_1 \in \mathcal{A}$. But, for $d = 3$, due to assumption (2.10), we also have

$$v_0,\ \Delta v_0,\ \Delta^2 v_0,\ v_1,\ \Delta v_1,\ \Delta^2 v_1 \in \mathcal{A},$$

which will be used to prove the decay of $Z$ with such initial conditions.

For $d = 2$, we observe that $v_1 \in \mathcal{A}$ entails that $\Delta v_1$, $\Delta^2 v_1 \in \mathcal{A}$ under the regularity assumptions on $v_1$ made in (2.8), (2.9). Indeed, a short computation in polar coordinates yields that

$$\Delta v_1(x) = \begin{cases} 0, & |\mathbf{x}| \leq \rho_0, \\ -\left(\frac{\omega}{c_0}\right)^2 \frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{|\mathbf{x}|^{3/2}} Y_1\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{5/2}}\right), & |\mathbf{x}| > \rho_1, \end{cases}$$

so $\Delta v_1 \in \mathcal{A}$ with $Y_w = -\left(\omega/c_0\right)^2 Y_1$. Iterating, we also obtain $\Delta^2 v_1 \in \mathcal{A}$ (with $Y_w = \left(\omega/c_0\right)^4 Y_1$). Since the assumptions on $v_0$ made in (2.8), (2.9) provide $v_0 \in \mathcal{A}$, we have similarly $\Delta v_0$, $\Delta^2 v_0 \in \mathcal{A}$.

Consequently, we deduce both for $d=2$ and $d=3$ that

$$\left|\Delta Z\left[0, v_1\right](\mathbf{x}, t)\right|=\left|Z\left[0, \Delta v_1\right](\mathbf{x}, t)\right| \leq \frac{C}{\left(1+t^2\right)^{1/2}}, \quad \mathbf{x} \in \Omega, \quad t \geq 0,$$

where the first equality is implied by $\partial_t^2 \Delta Z\left[0, v_1\right](\mathbf{x}, t)=\Delta \partial_t^2 Z\left[0, v_1\right](\mathbf{x}, t)$ which is due to $v_1 \in C^5\left(\mathbb{R}^d\right) \subset C^4\left(\mathbb{R}^d\right)$. Therefore, in view of validity of the wave equation, we deduce

$$\left|\partial_t^2 Z\left[0, v_1\right](\mathbf{x}, t)\right| \leq \frac{C}{\left(1+t^2\right)^{1/2}}, \quad \mathbf{x} \in \Omega, \quad t \geq 0.$$

For fixed $\mathbf{x} \in \Omega$, we interpolate between the decay estimates of $Z[0, v_1]$ and $\partial_t^2 Z[0, v_1]$ by using Lemma A.4. Then, we have

$$\left|\partial_t Z\left[0, v_1\right](\mathbf{x}, t)\right| \leq \frac{C}{\left(1+t^2\right)^{1/2}}, \quad \mathbf{x} \in \Omega, \quad t \geq 0.$$

We can write $Z\left[v_0, v_1\right]=Z\left[0, v_1\right]+\partial_t Z\left[0, v_0\right]$, and hence we have

$$\left|Z\left[v_0, v_1\right](\mathbf{x}, t)\right| \leq \frac{C}{\left(1+t^2\right)^{1/2}}, \quad \mathbf{x} \in \Omega, \quad t \geq 0, \tag{4.63}$$

due to $v_1$, $v_0$, $\Delta v_0 \in \mathcal{A}$.

Also, writing $\Delta Z\left[v_0, v_1\right]=Z\left[0, \Delta v_1\right]+\partial_t Z\left[0, \Delta v_0\right]$, we have

$$\left|\Delta Z\left[v_0, v_1\right](\mathbf{x}, t)\right| \leq \frac{C}{\left(1+t^2\right)^{1/2}}, \quad \mathbf{x} \in \Omega, \quad t \geq 0, \tag{4.64}$$

since $\Delta v_1$, $\Delta v_0$, $\Delta^2 v_0 \in \mathcal{A}$ and $v_0 \in C^6\left(\mathbb{R}^d\right) \subset C^5\left(\mathbb{R}^d\right)$, $v_1 \in C^5\left(\mathbb{R}^d\right) \subset C^4\left(\mathbb{R}^d\right)$. Using the wave equation, we interpolate between the decay estimates of $Z[v_0, v_1]$ and $\partial_t^2 Z[v_0, v_1]$ by employing Lemma A.4. We thus obtain

$$\left|\partial_t Z\left[v_0, v_1\right](\mathbf{x}, t)\right| \leq \frac{C}{\left(1+t^2\right)^{1/2}}, \quad \mathbf{x} \in \Omega, \quad t \geq 0. \tag{4.65}$$

Moreover, since $\Delta Z\left(0, \Delta v_0\right)=Z\left(0, \Delta^2 v_0\right)$ due to $v_0 \in C^6\left(\mathbb{R}^d\right)$, we can write

$$\begin{aligned} \partial_t \Delta Z\left[v_0, v_1\right] & =\partial_t Z\left[0, \Delta v_1\right]+\partial_t^2 Z\left[0, \Delta v_0\right] \\ & =\partial_t Z\left[0, \Delta v_1\right]+c_0^2 Z\left[0, \Delta^2 v_0\right]. \end{aligned}$$

Consequently,

$$\left|\partial_t \Delta Z\left[v_0, v_1\right](\mathbf{x}, t)\right| \leq \frac{C}{\left(1+t^2\right)^{1/2}}, \quad \mathbf{x} \in \Omega, \quad t \geq 0, \tag{4.66}$$

as $v_1$, $\Delta v_1$, $\Delta^2 v_1$, $\Delta^2 v_0 \in \mathcal{A}$ and $v_0 \in C^6\left(\mathbb{R}^d\right)$, $v_1 \in C^5\left(\mathbb{R}^d\right)$.

To deduce the estimates analogous to (4.64) and (4.66) but involving the gradient of $Z$ instead of the Laplacian, we use an interpolation argument. In particular, for a function $u \in C^2(\widetilde{\Omega})$, with some $\widetilde{\Omega}$ such that $\Omega \Subset \widetilde{\Omega}$, using interior elliptic regularity results, we obtain

$$\begin{aligned}\|\nabla u\|_{L^\infty(\Omega)} &\le \sup_{x\in\Omega}\left(\frac{1}{d_x}\right)\sup_{x\in\Omega}\left(d_x\left|\nabla u\left(x\right)\right|\right) \le C_{01}\sup_{x\in\widetilde{\Omega}}\left(d_x\left|\nabla u\left(x\right)\right|\right)\\ &\le C_{01}C_{02}\left[\|u\|_{L^\infty(\widetilde{\Omega})}+\sup_{x\in\widetilde{\Omega}} d_x^2\,\|\Delta u\|_{L^\infty(\widetilde{\Omega})}\right]\\ &\le C_{03}\left[\|u\|_{L^\infty(\widetilde{\Omega})}+\|\Delta u\|_{L^\infty(\widetilde{\Omega})}\right],\end{aligned} \tag{4.67}$$

with some constants $C_{01}, C_{02}, C_{03} > 0$. Here, we employed the notation $d_x := \operatorname{dist}\left(x, \partial\widetilde{\Omega}\right)$ and, in the second line, we used the first estimate of [16, Thm 3.9]. Observe now that, since in the statement of the present lemma, the domain $\Omega$ was arbitrary, all the previous steps of this proof remain valid (with different constants) for the larger domain $\widetilde{\Omega}$. In particular, estimates (4.63)–(4.66) are valid with $\Omega$ replaced by $\widetilde{\Omega}$. Consequently, according to (4.67) applied to $Z$ and to $\partial_t Z$ (permitted by the regularity (4.62)), we deduce

$$\left|\nabla Z\left[v_0, v_1\right](\mathbf{x}, t)\right| + \left|\partial_t \nabla Z\left[v_0, v_1\right](\mathbf{x}, t)\right| \le \frac{C}{\left(1+t^2\right)^{1/2}}, \quad \mathbf{x}\in\Omega, \quad t\ge 0,$$

and the proof is complete. □

### 4.4. A variant of Lemma 2.3.

In this subsection, we present a small variation of Lemma 2.3, which provides a weaker result under weaker assumptions. This lemma will be used as an auxiliary tool to prove Lemma 2.4.

**Lemma 4.1.** *Let $d = 2, 3$ and $\omega$, $\rho_0 > 0$, $\rho_1 > \rho_0$ be some fixed constants. Using the notation introduced in Lemma 2.3, suppose that $\Omega \Subset \mathbb{B}_{\rho_0}$. When $d = 2$, we assume that $v_0 \in C^1\left(\mathbb{R}^2\right)$, $v_1 \in C\left(\mathbb{R}^2\right)$ are such that*

$$|\mathbf{x}|^{5/2}\left(\left|v_0(\mathbf{x})\right| + \left|v_1(\mathbf{x})\right| + \left|\nabla v_0(\mathbf{x})\right|\right) \le C_0, \quad \mathbf{x}\in\mathbb{R}^2, \tag{4.68}$$

*and $v_0(\mathbf{x}) = v_1(\mathbf{x}) \equiv 0$ for $|\mathbf{x}| \le \rho_0$. When $d = 3$, we assume that $v_0 \in C^1\left(\mathbb{R}^3\right)$, $v_1 \in C\left(\mathbb{R}^3\right)$ are such that*

$$|\mathbf{x}|^{2}\left(\left|v_0(\mathbf{x})\right| + \left|v_1(\mathbf{x})\right| + \left|\nabla v_0(\mathbf{x})\right|\right) \le C_0, \quad \mathbf{x}\in\mathbb{R}^3. \tag{4.69}$$

*Then, there exists a constant $C > 0$ such that, for all $\mathbf{x} \in \Omega$ and $t \ge 0$, the solution of (2.2) satisfies*

$$\left|v(\mathbf{x}, t)\right| \le \frac{C}{\left(1+t^2\right)^{1/2}}, \quad \mathbf{x}\in\Omega, \quad t\ge 0. \tag{4.70}$$

*Proof.* We treat separately the cases $d = 2$ and 3.

- **Case** $d = 2$.

According to Poisson's formula [15, Par. 2.4.1 (c)] (see also (4.76) below), we have

$$v(\mathbf{x},t) = \frac{t}{2\pi}\int_0^1 \frac{r}{(1-r^2)^{1/2}}\int_{|\mathbf{s}|=1} v_1(\mathbf{x}+\mathbf{s}rc_0t)\,d\sigma_{\mathbf{s}}dr \\ + \frac{1}{2\pi}\int_0^1 \frac{r}{(1-r^2)^{1/2}}\int_{|\mathbf{s}|=1} v_0(\mathbf{x}+\mathbf{s}rc_0t)\,d\sigma_{\mathbf{s}}dr \\ + \frac{c_0t}{2\pi}\int_0^1 \frac{r^2}{(1-r^2)^{1/2}}\int_{|\mathbf{s}|=1} \mathbf{s}\cdot\nabla v_0(\mathbf{x}+\mathbf{s}rc_0t)\,d\sigma_{\mathbf{s}}dr \\ =: Q_1(\mathbf{x},t)+Q_2(\mathbf{x},t)+Q_3(\mathbf{x},t)\,. \tag{4.71}$$

Using the support assumption on $v_0$, we recall the definition of $a_1$ given by (4.26) and realise that the term $Q_1$ is identical to $Q$ in the proof of Lemma 2.3 (see (4.25)). Hence, (4.30) furnishes the required estimate of $Q_1$ due to assumption (4.68). The term $Q_2$ only differs from $Q_1$ by the absence of the $t$ factor and thus obeys an analogous estimate (in fact it is even $\mathcal{O}(t^{-2})$). Therefore, we immediately obtain

$$|Q_1(\mathbf{x},t)| + |Q_2(\mathbf{x},t)| \le \frac{C}{(1+t^2)^{1/2}}, \quad \mathbf{x}\in\Omega, \quad t \ge t_0, \tag{4.72}$$

with some constant $C>0$ and $t_0 = 2a_1$.

It thus remains to deal with $Q_3$. To this effect, we rewrite

$$Q_3(\mathbf{x},t) = \frac{c_0}{2\pi t^{3/2}}\int_{a_1/t}^1 \frac{1}{r^{1/2}(1-r^2)^{1/2}}\int_{|\mathbf{s}|=1}\left(\frac{rt}{\rho}\right)^{5/2}\rho^{5/2}\mathbf{s}\cdot\nabla v_0(\mathbf{x}+\mathbf{s}rc_0t)\,d\sigma_{\mathbf{s}}dr,$$

where $\rho := |\mathbf{x}+\mathbf{s}rc_0t|$, and we reduced the $r$-integration range, following the discussion "around" (4.26). Employing (4.29) and (4.68), we can estimate

$$|Q_3(\mathbf{x},t)| \le \frac{2^{5/2}C_0}{(c_0t)^{3/2}}\int_{a_1/t}^1 \frac{dr}{r^{1/2}(1-r^2)^{1/2}} \le \frac{2^{5/2}C_0}{c_0^{3/2}a_1^{1/2}}\frac{1}{t}\int_0^1\frac{dr}{(1-r^2)^{1/2}}$$

for $\mathbf{x}\in\Omega$ and $t\ge t_0$. Combined with (4.72), this furnishes the bound on $Q$. Continuity of $v$ (as follows from (4.71) due to the regularity assumptions on the initial data) implies that the bound can be extended to $t \ge 0$. Therefore, we conclude (4.70).

- **Case $d = 3$.**

Kirchhoff's formula [15, Par. 2.4.1 (c)] (see also (4.113) below) yields

$$v(\mathbf{x},t) = \frac{1}{4\pi}\int_{|\mathbf{s}|=1}\left[t\,v_1(\mathbf{x}+\mathbf{s}c_0t) + v_0(\mathbf{x}+\mathbf{s}c_0t) + tc_0\mathbf{s}\cdot\nabla v_0(\mathbf{x}+\mathbf{s}c_0t)\right]d\sigma_{\mathbf{s}}. \tag{4.73}$$

Hence, estimating each term in (4.73) using (4.29) and (4.69) directly implies (4.70). □

4.5. **Proof of Lemma 2.4.** Since by our assumptions $v_0 \in C^7(\mathbb{R}^d)$ and $v_1 \in C^6(\mathbb{R}^d)$, Theorems 2 and 3 in [15, Par. 2.4.1] (applied to $v$ and its derivatives) imply that the solution $v$ to (2.2) satisfies $v \in C^6(\mathbb{R}^d\times\mathbb{R}_+)$ and thus $\partial_t v \in C^5(\mathbb{R}^d\times\mathbb{R}_+)$.

Similarly to Lemma 2.3, in the main body of the proof, we shall prove the estimate

$$|v(\mathbf{x},t)| \le \frac{C}{(1+t^2)^{1/2}}, \quad \mathbf{x}\in\Omega, \ \ t\ge 0, \tag{4.74}$$

for some constant $C > 0$, and the estimate of the other terms in (2.13) is discussed at the end. Note that the estimate (4.74) can be proven under weaker regularity assumptions on $v_0$ and $v_1$ than those in the formulation of the present lemma but, on the other hand, it also holds true for a more general form of the initial conditions (allowing for the presence of faster decaying extra terms). This class of initial conditions shall be described in the definition of $\mathcal{B}$ given by (4.115).

We extend the function $Y$ from the sphere $\mathbb{S}^{d-1}$ to the spherical shell (annulus for $d = 2$) with the inner and outer radii $\frac{1}{2}$ and $\frac{3}{2}$, respectively. We choose a constant extension in the radial direction. This extension makes $\nabla Y$ well defined on $\mathbb{S}^{d-1}$. Moreover, $\nabla Y(\phi)$ is a tangent vector to $\mathbb{S}^{d-1}$ for each (radial vector) $\phi \in \mathbb{S}^{d-1}$, and hence

$$\phi \cdot \nabla Y(\phi) = 0. \tag{4.75}$$

- **Case $d = 2$.**

The solution is given explicitly by Poisson's formula [15, Par. 2.4.1 (c)]

$$\begin{aligned} v(\mathbf{x}, t) =& \frac{1}{2\pi}\left[\int_0^1 \frac{rt}{(1-r^2)^{1/2}} \int_{|\mathbf{s}|=1} v_1(\mathbf{x} + \mathbf{s}rc_0t)\, d\sigma_{\mathbf{s}} dr \right. \\ &\left. + \partial_t \left( \int_0^1 \frac{rt}{(1-r^2)^{1/2}} \int_{|\mathbf{s}|=1} v_0(\mathbf{x} + \mathbf{s}rc_0t)\, d\sigma_{\mathbf{s}} dr \right)\right]. \end{aligned} \tag{4.76}$$

Upon insertion of (2.12) into (4.76), and setting $\rho := |\mathbf{x} + \mathbf{s}rc_0t|$, $\phi := \frac{\mathbf{x}+\mathbf{s}rc_0t}{|\mathbf{x}+\mathbf{s}rc_0t|}$, we rearrange the terms to obtain

$$\begin{aligned} v(\mathbf{x}, t) =& \frac{1}{2\pi}\int_{|\mathbf{s}|=1}\int_0^1 \frac{rt}{(1-r^2)^{1/2}} \left[\partial_t A(\rho) - c_0 A'(\rho)\right] Y(\phi)\, dr d\sigma_{\mathbf{s}} \\ &+ \frac{1}{2\pi}\int_{|\mathbf{s}|=1}\int_0^1 \frac{r}{(1-r^2)^{1/2}} A(\rho)\left[t\partial_t Y(\phi) + Y(\phi)\right] dr d\sigma_{\mathbf{s}}. \end{aligned} \tag{4.77}$$

As $A(\rho) \equiv 0$ for $\rho \le \rho_0$, then $\phi$ is well-defined whenever it appears in the above integrands (i.e. $\rho = |\mathbf{x} + \mathbf{s}rc_0t| > \rho_0$ whenever $A(\rho) \ne 0$).

Here and in the sequel, we use the following notation to avoid having too many brackets: $\partial_t A(\rho) := \partial_t(A(\rho))$, $\partial_t Y(\phi) := \partial_t(Y(\phi))$, $\nabla Y(\phi) := (\nabla Y)(\phi)$.

Since $|\mathbf{s}| = 1$, we have

$$\partial_t A(\rho) = rc_0 A'(\rho)\frac{\mathbf{x}\cdot\mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|}, \qquad r\partial_r A(\rho) = t\partial_t A(\rho), \tag{4.78}$$

$$\partial_t Y(\phi) = \frac{rc_0\mathbf{s}\cdot\nabla Y(\phi)}{|\mathbf{x} + \mathbf{s}rc_0t|}, \qquad r\partial_r Y(\phi) = t\partial_t Y(\phi), \tag{4.79}$$

where we have used (4.75). Note that, using (4.78), we can rewrite, for $t > 0$,

$$\begin{aligned} \partial_t A(\rho) - c_0 A'(\rho) =& \frac{1}{t}(r-1)\partial_r A(\rho) - \frac{1}{t}\left(\frac{|\mathbf{x} + \mathbf{s}rc_0t|}{\mathbf{x}\cdot\mathbf{s} + rc_0t} - 1\right)\partial_r A(\rho) \\ =& \frac{1}{t}(r-1)\partial_r A(\rho) + c_0\left(\frac{\mathbf{x}\cdot\mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|} - 1\right)A'(\rho). \end{aligned} \tag{4.80}$$

Note that $\mathbf{x}\cdot\mathbf{s}+rc_0t$ cannot be zero on supp $A$ if $\mathbf{x}\in\Omega$ and $\mathbf{s}\in\mathbb{S}^1$. Indeed, $\mathbf{x}\cdot\mathbf{s}+rc_0t=0$ would imply

$$\rho^2 = |\mathbf{x}|^2 + 2rc_0t\mathbf{x}\cdot\mathbf{s} + r^2c_0^2t^2 = |\mathbf{x}|^2 - r^2c_0^2t^2 \le |\mathbf{x}|^2 < \rho_0^2,$$

and hence $A(\rho)=0$. Plugging (4.80) into (4.77), we observe that the term with $\partial_r A$ can be integrated by parts in the variable $r$, due to the cancellation of the singularity at $r=1$. In doing so, both boundary terms at $r=0$ and $r=1$, respectively, vanish. With some simplifications that employ (4.79) and the identity

$$\left(\frac{r\,(1-r)}{(1-r^2)^{1/2}}\right)' + \frac{r}{(1-r^2)^{1/2}} = \frac{1}{(1-r^2)^{1/2}\,(1+r)},$$

we arrive at

$$\begin{aligned} v(\mathbf{x},t) =& \frac{1}{2\pi}\int_{|\mathbf{s}|=1}\int_0^1 \frac{1}{(1-r^2)^{1/2}}\left[\frac{1}{1+r}A(\rho)\,Y(\phi)\right. \\ & \left. + rc_0t\left(\frac{\mathbf{x}\cdot\mathbf{s}+rc_0t}{|\mathbf{x}+\mathbf{s}rc_0t|}-1\right)A'(\rho)\,Y(\phi)\right]dr d\sigma_{\mathbf{s}} + Q(\mathbf{x},t)\,. \end{aligned} \tag{4.81}$$

Here, we have set

$$\begin{aligned} Q(\mathbf{x},t) :=& \frac{1}{2\pi}\int_{|\mathbf{s}|=1}\int_0^1 \frac{rc_0t}{(1-r^2)^{1/2}}\frac{1}{|\mathbf{x}+\mathbf{s}rc_0t|}A(\rho)\,\mathbf{s}\cdot\nabla Y(\phi)\,dr d\sigma_{\mathbf{s}} \\ =& -\frac{1}{2\pi}\int_{|\mathbf{s}|=1}\int_0^1 \frac{A(\rho)}{(1-r^2)^{1/2}}\frac{\mathbf{x}\cdot\nabla Y(\phi)}{|\mathbf{x}+\mathbf{s}rc_0t|}dr d\sigma_{\mathbf{s}}, \end{aligned} \tag{4.82}$$

where we used

$$\mathbf{x}\cdot\nabla Y(\phi) = -rc_0t\mathbf{s}\cdot\nabla Y(\phi), \tag{4.83}$$

which follows from (4.75).

For later reference, we note that, for $rt \gg 1$, uniformly in $\mathbf{x}\in\Omega$, $\mathbf{s}\in\mathbb{S}^1$,

$$-\frac{\mathbf{x}\cdot\nabla Y(\phi)}{|\mathbf{x}+\mathbf{s}rc_0t|} = \mathcal{O}\left(\frac{1}{rt}\right), \tag{4.84}$$

where we used $|\mathbf{x}|\le\rho_0$ and

$$\rho \ge |\mathbf{s}rc_0t| - |\mathbf{x}| \ge rc_0t - \rho_0 \ge \frac{1}{2}rc_0t \qquad \text{for } rt \ge \frac{2\rho_0}{c_0}.$$

By setting

$$P_1(\mathbf{x},t) := \frac{1}{2\pi}\int_0^1 \frac{1}{(1-r^2)^{1/2}}\int_{|\mathbf{s}|=1} rc_0t\left(\frac{\mathbf{x}\cdot\mathbf{s}+rc_0t}{|\mathbf{x}+\mathbf{s}rc_0t|}-1\right)A'(\rho)\,Y(\phi)\,d\sigma_{\mathbf{s}}dr, \tag{4.85}$$

$$P_2(\mathbf{x},t) := \frac{1}{2\pi}\int_0^1 \frac{1}{(1-r^2)^{1/2}\,(1+r)}\int_{|\mathbf{s}|=1} A(\rho)\,Y(\phi)\,d\sigma_{\mathbf{s}}dr, \tag{4.86}$$

we can rewrite (4.81) as

$$v(\mathbf{x},t) = P_1(\mathbf{x},t) + P_2(\mathbf{x},t) + Q(\mathbf{x},t)\,. \tag{4.87}$$

As in the proof of Lemma 2.3, we shall prove that there exists $t_0 > 0$ such that the bounds

$$|P_1(\mathbf{x},t)| \le \frac{\widetilde{C}_0}{t}, \quad |P_2(\mathbf{x},t)| \le \frac{\widetilde{C}_1}{t}, \quad |Q(\mathbf{x},t)| \le \frac{\widetilde{C}_2}{t}, \tag{4.88}$$

are valid uniformly in $\mathbf{x} \in \Omega$ with some constants $\widetilde{C}_0, \widetilde{C}_1, \widetilde{C}_2 > 0$ for any $t \ge t_0$. Then, due to the uniform boundedness of the solution $v$ on $\Omega \times [0, t_0]$ (see (4.76)), the bounds in (4.88) imply (4.74).

The functions $A(\rho)$ and $A'(\rho)$ may be different from zero only for $\rho > \rho_0$, i.e. for $rc_0t > \left((\mathbf{x}\cdot\mathbf{s})^2 + \rho_0^2 - |\mathbf{x}|^2\right)^{1/2} - \mathbf{x}\cdot\mathbf{s}$, obtained from $\rho^2 = (rc_0t + \mathbf{x}\cdot\mathbf{s})^2 - (\mathbf{x}\cdot\mathbf{s})^2 + |\mathbf{x}|^2 > \rho_0^2$. Thus, the integration range in the $r$ variable in each term of (4.87) effectively reduces from $(0,1)$ to $(a_1/t, 1)$ with $a_1 > 0$ defined in (4.26). With this argument, we are implicity assuming that $t \ge a_1$. We will actually prove (4.88) with $t_0 := 2a_1$.

Estimate of $P_1$ for $t \ge t_0$:
Let us introduce

$$F_1(\mathbf{x}, rt) := \frac{c_0}{2\pi} \int_{|\mathbf{s}|=1} (rt)^{5/2} \left( \frac{\mathbf{x}\cdot\mathbf{s} + rc_0t}{|\mathbf{x} + \mathbf{s}rc_0t|} - 1 \right) A'(\rho)\, Y(\phi)\, d\sigma_{\mathbf{s}}, \tag{4.89}$$

so that we can write

$$\begin{aligned} P_1(\mathbf{x},t) &= \int_{a_1/t}^{1} \frac{1}{(1-r^2)^{1/2}(rt)^{3/2}} F_1(\mathbf{x}, rt)\, dr = \int_{a_1/t}^{1/2} \dots + \int_{1/2}^{1} \dots \\ &=: P_{1,1}(\mathbf{x},t) + P_{1,2}(\mathbf{x},t), \end{aligned}$$

assuming $a_1/t \le 1/2$, i.e. $t \ge 2a_1$.

Since we consider only $\rho = |\mathbf{x} + \mathbf{s}rc_0t| > \rho_0 > 0$ (as the integrand of $F_1(\mathbf{x}, rt)$ vanishes otherwise), the denominators in (4.44) and (4.89) are bounded away from zero. Moreover, since

$$rc_0t = |\mathbf{s}rc_0t| \le \rho + |\mathbf{x}| \le \rho + \rho_0, \tag{4.90}$$

we have, uniformly for $\mathbf{x} \in \Omega$, $\mathbf{s} \in \mathbb{S}^1$,

$$(rt)^{1/2} |A'(\rho)| \le \begin{cases} \left(\frac{\rho+\rho_0}{c_0}\right)^{1/2} \|A'\|_{L^\infty(\mathbb{R}_+)}, & 0 \le \rho \le \rho_1, \\ \left(\frac{rt}{\rho}\right)^{1/2}\left(\frac{\omega}{c_0} + \frac{1}{2\rho_1}\right) \le \left(\frac{\rho+\rho_0}{c_0\rho}\right)^{1/2}\left(\frac{\omega}{c_0} + \frac{1}{2\rho_1}\right), & \rho > \rho_1, \end{cases} \tag{4.91}$$
$$\le C_0, \quad \rho > 0,$$

for some constant $C_0 > 0$. This, together with (4.44), implies that

$$\sup_{\mathbf{x}\in\Omega} \|F_1(\mathbf{x}, \cdot)\|_{L^\infty(a_1,\infty)} =: C_1 < \infty,$$

for some constant $C_1 > 0$. Therefore, we can estimate, for $t \ge t_0 = 2a_1$ and $\mathbf{x} \in \Omega$,

$$|P_{1,1}(\mathbf{x},t)| \le 2^{1/2} \frac{C_1}{t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} < 2^{1/2} \frac{C_1}{t^{3/2}} \int_{a_1/t}^{\infty} \frac{dr}{r^{3/2}} = \frac{2^{3/2} C_1}{a_1^{1/2} t},$$

$$|P_{1,2}(\mathbf{x},t)| \le 2^{3/2} \frac{C_1}{t^{3/2}} \int_{1/2}^{1} \frac{dr}{(1-r)^{1/2}} = \frac{4C_1}{t^{3/2}},$$

and thus we have the bound for $P_1$ in (4.88) with $t_0 = 2a_1$ and some constant $\widetilde{C}_0 > 0$.

Estimate of $Q$ for $t \geq t_0$:

As above, we note that all the denominators in (4.82) are bounded away from zero. Therefore, by setting

$$F_2(\mathbf{x}, rt) := -\frac{(rt)^{3/2}}{2\pi} \int_{|\mathbf{s}|=1} \frac{\mathbf{x} \cdot \nabla Y(\phi)}{|\mathbf{x} + \mathbf{s} r c_0 t|} A(\rho)\, d\sigma_{\mathbf{s}}, \tag{4.92}$$

and recalling (4.84) and (4.90), we have

$$\sup_{\mathbf{x} \in \Omega} \|F_2(\mathbf{x}, \cdot)\|_{L^\infty(a_1, \infty)} =: C_2 < \infty. \tag{4.93}$$

Consequently, we estimate as before, for $\mathbf{x} \in \Omega$, $t \geq t_0 = 2a_1$,

$$|Q(\mathbf{x}, t)| \leq 2^{1/2} \frac{C_2}{t^{3/2}} \int_{a_1/t}^{1/2} \frac{dr}{r^{3/2}} + 2^{3/2} \frac{C_2}{t^{3/2}} \int_{1/2}^{1} \frac{dr}{(1-r)^{1/2}} < \frac{\widetilde{C}_2}{t},$$

with some constant $\widetilde{C}_2 > 0$. This proves the bound for $Q$ in (4.88) again with $t_0 = 2a_1$.

Estimate of $P_2$ for $t \geq t_0$:

For the term $P_2$, we proceed as in the estimate of the term $P$ in Lemma 2.3. Let us rewrite (4.86) as

$$\begin{aligned} P_2(\mathbf{x}, t) =& \frac{1}{2\pi} \int_0^1 \frac{e^{ir\omega t}}{(1-r)^{1/2}} \int_{|\mathbf{s}|=1} \left[ \frac{e^{-ir\omega t}}{(1+r)^{3/2}} A(\rho)\, Y(\phi) \right. \\ & \left. - \frac{e^{-i\omega t}}{2^{3/2}} A(|\mathbf{x} + \mathbf{s} c_0 t|)\, Y\left( \frac{\mathbf{x} + \mathbf{s} c_0 t}{|\mathbf{x} + \mathbf{s} c_0 t|} \right) \right] d\sigma_{\mathbf{s}} dr \\ & + \frac{1}{2^{5/2}\pi} \int_{|\mathbf{s}|=1} A(|\mathbf{x} + \mathbf{s} c_0 t|)\, Y\left( \frac{\mathbf{x} + \mathbf{s} c_0 t}{|\mathbf{x} + \mathbf{s} c_0 t|} \right) d\sigma_{\mathbf{s}} \int_0^1 \frac{e^{-i(1-r)\omega t}}{(1-r)^{1/2}} dr \\ =:& P_{2,1}(\mathbf{x}, t) + P_{2,2}(\mathbf{x}, t). \end{aligned} \tag{4.94}$$

We start with

$$P_{2,2}(\mathbf{x}, t) = \frac{1}{t^{1/2}} F_4(\mathbf{x}, t) \int_0^1 \frac{e^{-ir\omega t}}{r^{1/2}} dr,$$

where we made a change of variable $r \mapsto (1-r)$ and introduced

$$F_4(\mathbf{x}, t) := \frac{t^{1/2}}{2^{5/2}\pi} \int_{|\mathbf{s}|=1} A(|\mathbf{x} + \mathbf{s} c_0 t|)\, Y\left( \frac{\mathbf{x} + \mathbf{s} c_0 t}{|\mathbf{x} + \mathbf{s} c_0 t|} \right) d\sigma_{\mathbf{s}}. \tag{4.95}$$

Similarly to (4.91), but with setting $r = 1$, we have $t^{1/2} A(|\mathbf{x} + \mathbf{s} c_0 t|) \leq C_{00}$ for some constant $C_{00} > 0$, and thus

$$\sup_{\mathbf{x} \in \Omega} \|F_4(\mathbf{x}, \cdot)\|_{L^\infty(\mathbb{R}_+)} =: C_4 < \infty. \tag{4.96}$$

Hence, employing Lemma A.3, we obtain uniformly for $\mathbf{x} \in \Omega$ and sufficiently large $t > 0$,

$$|P_{2,2}(\mathbf{x}, t)| \leq \frac{\widetilde{C}_4}{t}. \tag{4.97}$$

Decay of $P_{2,1}$. To deal with $P_{2,1}$, we note that the integrand is a smooth function of $r$ in $[0,1)$ and it behaves like $(1-r)^{1/2}$ as $r \to 1$. Integrating by parts in the $r$ variable with $e^{ir\omega t}dr$ as differential, both boundary terms vanish (recall also that $A(|\mathbf{x}|) \equiv 0$ for $\mathbf{x} \in \Omega$). We thus have

$$\begin{aligned} P_{2,1}(\mathbf{x},t) = & -\frac{1}{2\pi i\omega t}\int_0^1 \int_{|\mathbf{s}|=1} e^{ir\omega t}\partial_r\left(\frac{1}{(1-r)^{1/2}}\left[\frac{e^{-ir\omega t}}{(1+r)^{3/2}}A(\rho)Y(\phi)\right.\right. \\ & \left.\left. -\frac{e^{-i\omega t}}{2^{3/2}}A(|\mathbf{x}+\mathbf{s}c_0 t|)Y\left(\frac{\mathbf{x}+\mathbf{s}c_0 t}{|\mathbf{x}+\mathbf{s}c_0 t|}\right)\right]\right) d\sigma_{\mathbf{s}}dr \\ = & I_1(\mathbf{x},t) + I_2(\mathbf{x},t) + I_3(\mathbf{x},t) + I_4(\mathbf{x},t), \end{aligned} \tag{4.98}$$

where

$$I_1(\mathbf{x},t) := \frac{i}{2\pi\omega t}\int_0^1 \int_{|\mathbf{s}|=1} \frac{1}{(1-r)^{1/2}(1+r)^{3/2}} Y(\phi)\, e^{ir\omega t}\partial_r\left(e^{-ir\omega t}A(\rho)\right) d\sigma_{\mathbf{s}}dr, \tag{4.99}$$

$$I_2(\mathbf{x},t) := \frac{i}{2\pi\omega t}\int_0^1 \int_{|\mathbf{s}|=1} \frac{1}{(1-r)^{1/2}(1+r)^{3/2}} A(\rho)\,\partial_r Y(\phi)\, d\sigma_{\mathbf{s}}dr, \tag{4.100}$$

$$I_3(\mathbf{x},t) := -\frac{3i}{4\pi\omega t}\int_0^1 \int_{|\mathbf{s}|=1} \frac{1}{(1-r)^{1/2}(1+r)^{5/2}} A(\rho)\,Y(\phi)\, d\sigma_{\mathbf{s}}dr, \tag{4.101}$$

$$\begin{aligned} I_4(\mathbf{x},t) := & \frac{i}{4\pi\omega t}\int_0^1 \int_{|\mathbf{s}|=1} \frac{1}{(1-r)^{3/2}}\left[\frac{1}{(1+r)^{3/2}}A(\rho)Y(\phi)\right. \\ & \left. -\frac{e^{-i(1-r)\omega t}}{2^{3/2}}A(|\mathbf{x}+\mathbf{s}c_0 t|)Y\left(\frac{\mathbf{x}+\mathbf{s}c_0 t}{|\mathbf{x}+\mathbf{s}c_0 t|}\right)\right] d\sigma_{\mathbf{s}}dr. \end{aligned} \tag{4.102}$$

Decay of $I_1$. For the term $I_1$ observe that, using (4.78), we have

$$\begin{aligned} \frac{r^{3/2}t^{1/2}}{c_0}e^{ir\omega t}\partial_r\left(e^{-ir\omega t}A(\rho)\right) = & (rt)^{3/2}\left(A'(\rho) - \frac{i\omega}{c_0}A(\rho)\right) \\ & -(rt)^{3/2}A'(\rho)\left(1 - \frac{\mathbf{x}\cdot\mathbf{s} + rc_0 t}{|\mathbf{x}+\mathbf{s}rc_0 t|}\right), \end{aligned}$$

where both terms on the right-hand side are uniformly bounded for $rt > 0$, $\mathbf{x} \in \Omega$, $|\mathbf{s}| = 1$, due to (4.44), (4.90), and the assumption on the form of $A$. Therefore, for

$$F_5(\mathbf{x}, rt) := \frac{ic_0}{2\pi\omega}\int_{|\mathbf{s}|=1} Y(\phi)\,\frac{r^{3/2}t^{1/2}}{c_0}e^{ir\omega t}\partial_r\left(e^{-ir\omega t}A(\rho)\right) d\sigma_{\mathbf{s}},$$

we have

$$\sup_{\mathbf{x}\in\Omega}\|F_5(\mathbf{x},\cdot)\|_{L^\infty(a_1,\infty)} =: C_5 < \infty.$$

Since both $A$, $A'$ vanish on $[0,\rho_0]$, the interval of integration for the $r$-integrals in each of (4.99)–(4.101) reduces to $(a_1/t,1)$ (see the discussion before (4.26)). Hence we can estimate $I_1$ in (4.99) for $t\geq t_0=2a_1$ as

$$\begin{aligned}|I_1(\mathbf{x},t)| =& \frac{1}{t^{3/2}}\left|\int_{a_1/t}^{1}\frac{1}{r^{3/2}(1-r)^{1/2}(1+r)^{3/2}}F_5(\mathbf{x},rt)\,dr\right| \\ \leq& \frac{2^{1/2}C_5}{t^{3/2}}\int_{a_1/t}^{1/2}\frac{dr}{r^{3/2}}+\frac{2^{3/2}C_5}{t^{3/2}}\int_{1/2}^{1}\frac{dr}{(1-r)^{1/2}}\leq\frac{\widetilde{C}_5}{t}\end{aligned} \tag{4.103}$$

with some constant $\widetilde{C}_5>0$.

Decay of $I_2$, $I_3$. In a similar but simpler fashion, we can estimate the terms $I_2$ and $I_3$. To this end we introduce

$$F_6(\mathbf{x},rt):=-\frac{3i(rt)^{1/2}}{4\pi\omega}\int_{|\mathbf{s}|=1}A(\rho)\,Y(\phi)\,d\sigma_{\mathbf{s}},$$

which satisfies

$$|F_6(\mathbf{x},rt)|\leq\frac{3}{4\pi\omega}\int_{|\mathbf{s}|=1}\left(\frac{rt}{\rho}\right)^{1/2}\rho^{1/2}\,|A(\rho)|\,|Y(\phi)|\,d\sigma_{\mathbf{s}},$$

and thus

$$\sup_{\mathbf{x}\in\Omega}\|F_6(\mathbf{x},\cdot)\|_{L^\infty(\mathbb{R}_+)}=:C_6<\infty. \tag{4.104}$$

Then, using (4.93) and (4.104), we obtain for $t\geq t_0=2a_1$,

$$\begin{aligned}|I_2(\mathbf{x},t)| =& \frac{1}{\omega t^{5/2}}\left|\int_{a_1/t}^{1}\frac{1}{r^{5/2}(1-r)^{1/2}(1+r)^{3/2}}F_2(\mathbf{x},rt)\,dr\right| \\ \leq& \frac{2^{1/2}C_2}{\omega t^{5/2}}\int_{a_1/t}^{1/2}\frac{dr}{r^{5/2}}+\frac{2^{5/2}C_2}{\omega t^{5/2}}\int_{1/2}^{1}\frac{dr}{(1-r)^{1/2}}\leq\frac{\bar{C}_2}{t},\end{aligned} \tag{4.105}$$

$$\begin{aligned}|I_3(\mathbf{x},t)| =& \frac{1}{t^{3/2}}\left|\int_{a_1/t}^{1}\frac{1}{r^{1/2}(1-r)^{1/2}(1+r)^{5/2}}F_6(\mathbf{x},rt)\,dr\right| \\ \leq& \frac{2^{1/2}C_6}{t^{3/2}}\left(\int_{a_1/t}^{1/2}\frac{dr}{r^{1/2}}+\int_{1/2}^{1}\frac{dr}{(1-r)^{1/2}}\right)\leq\frac{\widetilde{C}_6}{t^{3/2}}\end{aligned} \tag{4.106}$$

with some constants $\bar{C}_2$, $\widetilde{C}_6>0$.

Decay of $I_4$. To treat the term $I_4$, we introduce

$$\begin{aligned}\widetilde{F}_7(\mathbf{x},r,t):=\int_{|\mathbf{s}|=1}\Bigg[&\frac{1}{(1+r)^{3/2}}A(\rho)\,Y(\phi) \\ &-\frac{e^{-i(1-r)\omega t}}{2^{3/2}}A(|\mathbf{x}+\mathbf{s}c_0t|)\,Y\left(\frac{\mathbf{x}+\mathbf{s}c_0t}{|\mathbf{x}+\mathbf{s}c_0t|}\right)\Bigg]\,d\sigma_{\mathbf{s}},\end{aligned} \tag{4.107}$$

$$F_7(\mathbf{x},r,t) := \frac{1}{1-r}\widetilde{F}_7(\mathbf{x},r,t)\,. \tag{4.108}$$

Using $\widetilde{F}_7(\mathbf{x},1,t)=0$, (4.108) can be rewritten as

$$F_7(\mathbf{x},r,t) = -\frac{1}{1-r}\left(\widetilde{F}_7(\mathbf{x},1,t)-\widetilde{F}_7(\mathbf{x},r,t)\right) = -\frac{1}{1-r}\int_r^1 \partial_r \widetilde{F}_7(\mathbf{x},\tau,t)\,d\tau. \tag{4.109}$$

From this we estimate for $r\in(1-a_1/t,1)$,

$$\begin{aligned}|F_7(\mathbf{x},r,t)| \le& \left\|\partial_r\widetilde{F}_7(\mathbf{x},\cdot,t)\right\|_{L^\infty(1-a_1/t,1)}\\ \le& \sup_{r\in(1-a_1/t,\,1)}\int_{|\mathbf{s}|=1}\left[\frac{3}{2(1+r)^{5/2}}|A(\rho)|\,|Y(\phi)| + \frac{1}{(1+r)^{3/2}}|\partial_r A(\rho)|\,|Y(\phi)|\right.\\ &\left. + \frac{1}{(1+r)^{3/2}}|A(\rho)|\,|\partial_r Y(\phi)| + \frac{\omega t}{2^{3/2}}|A(|\mathbf{x}+\mathbf{s}c_0t|)|\left|Y\left(\frac{\mathbf{x}+\mathbf{s}c_0t}{|\mathbf{x}+\mathbf{s}c_0t|}\right)\right|\right]d\sigma_{\mathbf{s}}.\end{aligned}$$

Therefore, employing (4.90), (4.78) and (4.79), taking into account the behaviour of $A(\rho)$ and $A'(\rho)$ for $\rho>\rho_1$, we deduce that

$$\sup_{\mathbf{x}\in\Omega,\ r\in(1-a_1/t,\,1),\ t>2a_1}\left|F_7(\mathbf{x},r,t)/t^{1/2}\right| =: C_7<\infty. \tag{4.110}$$

Moreover, for $r\in(1/2,1-a_1/t)$ and $\epsilon\in(0,1/2]$, taking into account (4.108), we have

$$\left|(1-r)^{1/2+\epsilon}\,t^\epsilon F_7(\mathbf{x},r,t)\right| = \left|\widetilde{F}_7(\mathbf{x},r,t)\right|\frac{t^\epsilon}{(1-r)^{1/2-\epsilon}} \le \left|\widetilde{F}_7(\mathbf{x},r,t)\right|\frac{t^{1/2}}{a_1^{1/2-\epsilon}}.$$

Hence, with $C_4$ and $C_6$ defined in (4.96) and (4.104), respectively, we obtain from (4.107) that, for any $\epsilon\in(0,1/2]$,

(4.111)

$$\begin{aligned}\sup_{\mathbf{x}\in\Omega,\ r\in(1/2,\,1-a_1/t),\ t>2a_1}\left|(1-r)^{1/2+\epsilon}\,t^\epsilon F_7(\mathbf{x},r,t)\right| \le& \frac{1}{a_1^{1/2-\epsilon}}\left(\frac{2^{1/2}4\pi\omega}{3}C_6+2\pi C_4\right)\\ =:& C_8<\infty.\end{aligned}$$

Altogether (4.111) and (4.110) imply that, for $t\ge t_0=2a_1$ and $\epsilon\in(0,1/2]$,

$$\begin{aligned}|I_4(\mathbf{x},t)| \le& \frac{1}{4\pi\omega t}\int_0^1\frac{1}{(1-r)^{1/2}}|F_7(\mathbf{x},r,t)|\,dr\\ =& \frac{1}{4\pi\omega t}\left(\int_0^{1/2}\ldots+\int_{1/2}^{1-a_1/t}\ldots+\int_{1-a_1/t}^1\ldots\right)\\ \le& \frac{2^{3/2}}{t^{3/2}}\left(\frac{C_6}{3}\int_0^{1/2}\frac{dr}{r^{1/2}}+\frac{C_4}{4\omega}\right)+\frac{C_8}{4\pi\omega t^{1+\epsilon}}\int_{1/2}^{1-a_1/t}\frac{dr}{(1-r)^{1+\epsilon}}\\ &+\frac{C_7}{4\pi\omega t^{1/2}}\int_{1-a_1/t}^1\frac{dr}{(1-r)^{1/2}}\le\frac{\widetilde{C}_7}{t}\end{aligned} \tag{4.112}$$

with some constant $\widetilde{C}_7 > 0$. Here, for the interval $(0, 1/2)$, we have estimated the integrand of $F_7$ directly, using again (4.104) and (4.96).

From estimates (4.103), (4.105), (4.106), and (4.112), of $I_1$, $I_2$, $I_3$, and $I_4$, respectively, in decomposition (4.98), we obtain for $P_{2,1}$ the same estimate as (4.97) for $P_{2,2}$.

The estimate for $P_2$ in (4.88) with $t_0 = 2a_1$ readily follows from (4.94). This completes the proof of (4.74) in the case $d = 2$.

- **Case $d = 3$.**

In this case, the solution is given by Kirchhoff's formula [15, Par. 2.4.1 (c)]

$$v(\mathbf{x}, t) = \frac{t}{4\pi} \int_{|\mathbf{s}|=1} \left[ v_1(\mathbf{x} + \mathbf{s}c_0 t) + \partial_t v_0(\mathbf{x} + \mathbf{s}c_0 t) + \frac{1}{t} v_0(\mathbf{x} + \mathbf{s}c_0 t) \right] d\sigma_{\mathbf{s}}. \tag{4.113}$$

The assumed form of the initial conditions yields

(4.114)

$$\begin{aligned} v(\mathbf{x}, t) =& \frac{t}{4\pi} \int_{|\mathbf{s}|=1} \left[ \left( \frac{\mathbf{x}\cdot\mathbf{s} + c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|} - 1 \right) c_0 A'(|\mathbf{x} + \mathbf{s}c_0 t|)\, Y\left( \frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|} \right) \right. \\ & \left. + A(|\mathbf{x} + \mathbf{s}c_0 t|)\, \partial_t Y\left( \frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|} \right) + \frac{1}{t} A(|\mathbf{x} + \mathbf{s}c_0 t|)\, Y\left( \frac{\mathbf{x} + \mathbf{s}c_0 t}{|\mathbf{x} + \mathbf{s}c_0 t|} \right) \right] d\sigma_{\mathbf{s}}. \end{aligned}$$

For $d = 3$, both $A(|\mathbf{x} + \mathbf{s}c_0 t|)$ and $A'(|\mathbf{x} + \mathbf{s}c_0 t|)$ are $\mathcal{O}(1/t)$ for $t \gg 1$ uniformly for $\mathbf{x} \in \Omega$, $\mathbf{s} \in \mathbb{S}^2$ (due to (4.90) with $r = 1$). Hence, the last term in the integrand is $\mathcal{O}(1/t^2)$ and employing (4.44) and (4.79) (both for $r = 1$), we observe that the first and the second terms are $\mathcal{O}(1/t^3)$ and $\mathcal{O}(1/t^2)$, respectively. Therefore, the whole integrand is $\mathcal{O}(1/t^2)$. The estimate (4.74) hence follows.

- **Conclusion of the proof.**

Given a pair of functions $(w_0, w_1)$ and constants $\omega$, $c_0$, $\rho_0 > 0$, $\rho_1 > \rho_0$, we say that

$$(w_0, w_1) \in \mathcal{B}, \tag{4.115}$$

if we can write

$$w_0(\mathbf{x}) = A_w(|\mathbf{x}|) Y_w\left( \frac{\mathbf{x}}{|\mathbf{x}|} \right) + V_w^{(1)}(\mathbf{x}), \tag{4.116}$$

$$w_1(\mathbf{x}) = -c_0 A_w'(|\mathbf{x}|) Y_w\left( \frac{\mathbf{x}}{|\mathbf{x}|} \right) + V_w^{(2)}(\mathbf{x}) \tag{4.117}$$

for some functions

$$A_w \in C^1(\mathbb{R}_+), \quad Y_w \in C^1(\mathbb{S}^{d-1}), \quad V_w^{(1)} \in C^1(\mathbb{R}^d), \quad V_w^{(2)} \in C(\mathbb{R}^d) \tag{4.118}$$

such that

$$A_w(|\mathbf{x}|) = V_w^{(1)}(\mathbf{x}) = V_w^{(2)}(\mathbf{x}) \equiv 0, \quad |\mathbf{x}| \le \rho_0, \qquad A_w(\rho) = \frac{e^{i\frac{\omega}{c_0}\rho}}{\rho^{\frac{d-1}{2}}}, \quad \rho > \rho_1,$$

and we have

$$|\mathbf{x}|^{\frac{d+3}{2}} \left( |V_w^{(1)}(\mathbf{x})| + |V_w^{(2)}(\mathbf{x})| + |\nabla V_w^{(1)}(\mathbf{x})| \right) \le C, \quad \mathbf{x} \in \mathbb{R}^d,$$

for some constant $C > 0$.

Let $Z[w_0, w_1] \equiv Z_w$ denote the solution of the wave equation $\partial_t^2 Z_w(\mathbf{x}, t) - c_0^2 \Delta Z_w(\mathbf{x}, t) = 0$ for $\mathbf{x} \in \mathbb{R}^d$, $t > 0$, subject to the initial conditions $Z_w(\mathbf{x}, 0) = w_0(\mathbf{x})$, $\partial_t Z_w(\mathbf{x}, 0) = w_1(\mathbf{x})$. By linearity, we have

$$Z_w = Z_{AY} + Z_V, \tag{4.119}$$

where the first term corresponds to the solution produced by the $A_w Y_w$ terms whereas the second one is due to the $V_w^{(1)}$, $V_w^{(2)}$ terms in (4.116), (4.117).

Note that, in the proof of the present lemma, we have already shown the decay

$$|Z_{AY}(\mathbf{x}, t)| \leq \frac{C}{(1+t^2)^{1/2}}, \quad t \geq 0, \quad \mathbf{x} \in \Omega,$$

since the regularity of $A$ and $Y$ in (4.118) was sufficient for this decay. An analogous time-decay estimate holds for the $Z_V$ term in (4.119), as follows from Lemma 4.1. Consequently, we obtain

$$|Z_w(\mathbf{x}, t)| \leq \frac{C}{(1+t^2)^{1/2}}, \quad t \geq 0, \quad \mathbf{x} \in \Omega.$$

In other words, class (4.115) consists of the initial conditions with somewhat minimal assumptions for which we can deduce the $\mathcal{O}(1/t)$ decay of the solution (but not necessarily of its derivatives).

Now we consider $Z[v_0, v_1] \equiv: Z$. Clearly, we have $(v_0, v_1) \in \mathcal{B}$ with $V_w^{(1)} = V_w^{(2)} \equiv 0$ and thus

$$|Z(\mathbf{x}, t)| \leq \frac{C}{(1+t^2)^{1/2}}, \quad t \geq 0, \quad \mathbf{x} \in \Omega. \tag{4.120}$$

We shall deduce similar results for $\Delta Z$ and $\Delta^2 Z$. To this effect, we first compute, for $\rho > \rho_1$,

$$A'(\rho) = \left(\frac{i\omega}{c_0} - \frac{d-1}{2\rho}\right) \frac{e^{i\frac{\omega}{c_0}\rho}}{\rho^{\frac{d-1}{2}}},$$

$$A''(\rho) = -\left(\frac{\omega^2}{c_0^2} + i\frac{\omega}{c_0}\frac{d-1}{\rho}\right) \frac{e^{i\frac{\omega}{c_0}\rho}}{\rho^{\frac{d-1}{2}}} + \mathcal{O}\left(\frac{1}{\rho^{\frac{d+3}{2}}}\right),$$

$$A'''(\rho) = -\frac{\omega^2}{c_0^2}\left(i\frac{\omega}{c_0} - \frac{3(d-1)}{2\rho}\right) \frac{e^{i\frac{\omega}{c_0}\rho}}{\rho^{\frac{d-1}{2}}} + \mathcal{O}\left(\frac{1}{\rho^{\frac{d+3}{2}}}\right).$$

Therefore, we have for $|\mathbf{x}| > \rho_1$ (with a computation in polar/spherical coordinates):

(4.121)

$$\begin{aligned}\Delta v_0(\mathbf{x}) &= A''(|\mathbf{x}|)\, Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + \frac{d-1}{|\mathbf{x}|} A'(|\mathbf{x}|)\, Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + \frac{1}{|\mathbf{x}|^2} A(|\mathbf{x}|)\, \Delta_{\mathbb{S}^{d-1}} Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \\ &= -\frac{\omega^2}{c_0^2} \frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{|\mathbf{x}|^{\frac{d-1}{2}}} Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right),\end{aligned}$$

(4.122)

$$-\frac{1}{c_0}\Delta v_1(\mathbf{x}) = A'''(|\mathbf{x}|)\,Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + \frac{d-1}{|\mathbf{x}|}A''(|\mathbf{x}|)\,Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + \frac{1}{|\mathbf{x}|^2}A'(|\mathbf{x}|)\,\Delta_{\mathbb{S}^{d-1}}Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)$$

$$= -\frac{\omega^2}{c_0^2}\left(i\frac{\omega}{c_0} - \frac{d-1}{2|\mathbf{x}|}\right)\frac{e^{i\frac{\omega}{c_0}|\mathbf{x}|}}{|\mathbf{x}|^{\frac{d-1}{2}}}Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right),$$

where $\Delta_{\mathbb{S}^{d-1}}$ is the Laplace-Beltrami operator on the $(d-1)$-dimensional unit sphere. Note that, due to cancellation, we do not have any $\mathcal{O}\left(1/|\mathbf{x}|^{\frac{d+1}{2}}\right)$ terms in (4.121). Also, observe that we can rewrite these quantities as

$$\Delta v_0(\mathbf{x}) = -\left(\frac{\omega}{c_0}\right)^2 A(|\mathbf{x}|)\,Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right),$$

$$\Delta v_1(\mathbf{x}) = \left(\frac{\omega}{c_0}\right)^2 c_0 A'(|\mathbf{x}|)\,Y\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right),$$

respectively. This, together with the regularity assumptions of this lemma, yields that $(\Delta v_0, \Delta v_1) \in \mathcal{B}$ (with $Y_w = -(\omega/c_0)^2 Y$ and some $V_w^{(1)}$, $V_w^{(2)}$ which are no longer identically zero). Since $\Delta Z[v_0, v_1] = Z[\Delta v_0, \Delta v_1]$ (implied by $Z \in C^6\left(\mathbb{R}^d \times \mathbb{R}_+\right)$ due to $v_0 \in C^7\left(\mathbb{R}^d\right)$ and $v_1 \in C^6\left(\mathbb{R}^d\right)$, as discussed in an analogous setting in the Conclusion of the proof of Lemma 2.3), we obtain

$$|\Delta Z(\mathbf{x}, t)| \le \frac{C}{(1+t^2)^{1/2}}, \quad t \ge 0, \quad \mathbf{x} \in \Omega. \tag{4.123}$$

By iteration we also deduce that $\left(\Delta^2 v_0, \Delta^2 v_1\right) \in \mathcal{B}$ and, using again the regularity $Z \in C^6\left(\mathbb{R}^d \times \mathbb{R}_+\right)$, that $\Delta^2 Z = Z\left[\Delta^2 v_0, \Delta^2 v_1\right]$. Consequently,

(4.124)

$$\left|\Delta^2 Z(\mathbf{x}, t)\right| = \frac{1}{c_0^2}\left|\partial_t^2 \Delta Z(\mathbf{x}, t)\right| = \frac{1}{c_0^4}\left|\partial_t^4 Z(\mathbf{x}, t)\right| \le \frac{C}{(1+t^2)^{1/2}}, \quad t \ge 0, \quad \mathbf{x} \in \Omega.$$

Employing the interpolation argument as at the final stage of the proof of Lemma 2.3, we use (4.120), (4.123), and (4.124) to obtain the bounds for all the intermediate derivatives in space and time, thus deducing (2.13). □

## 5. Conclusions and outlook

Motivated by the development of time-domain methods for the numerical solution of Helmholtz problems with variable coefficients, we have established a rigorous proof of the LAP under physically reasonable assumptions on the coefficients of the wave equation and the source term. Under an appropriate modification, the LAP has been extended to 1D. Moreover, since the speed of stabilisation towards the harmonic regime is a deciding factor for using time-domain approaches in practice, we have also provided rigorous estimates for this convergence in time.

Our main focus was on the 1D and 2D cases for which the LAP was generally understudied previously. In these cases, exponential (for 1D) and algebraic (for 2D) convergence rates are generally sharp. In the 3D case, previous works on wave equations of similar form and some of our numerical experiments (for radial data, see Appendix B) seem to suggest that our algebraic convergence result could be improved to the exponential one.

An interesting extension of our results would be to remove the non-trapping assumption on the coefficients. Even though the LAP is still expected to be valid, in this case, the time convergence rate would be much slower. Namely, [31, Thm. 3] suggests that the convergence rate $1/(1+t)$ would be replaced by $1/[\log(2+t)]^{\gamma}$ for some $\gamma > 0$.

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## Appendix A.

We collect here some technical estimates needed in the proofs of Sections 3–4.

**Lemma A.1.** *For $\mathbf{y} \in \Omega$, a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, and $K$ defined by (3.8), the following asymptotic expansions are valid for $|\mathbf{x}| \gg 1$:*

$$K(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi}\left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}} \frac{e^{i\frac{\omega}{c_0}\left(|\mathbf{x}|-\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)}}{|\mathbf{x}|^{\frac{d-1}{2}}}\left[1+\frac{1}{|\mathbf{x}|}\left((d-3)(d-1)\frac{ic_0}{8\omega} + \frac{d-1}{2}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} + \frac{i\omega}{2c_0}\frac{|\mathbf{x}|^2|\mathbf{y}|^2-(\mathbf{x}\cdot\mathbf{y})^2}{|\mathbf{x}|^2}\right)\right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right), \tag{A.1}$$

$$\partial_{|\mathbf{x}|}K(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi}\left(\frac{\omega}{2\pi i c_0}\right)^{\frac{d-3}{2}} \frac{e^{i\frac{\omega}{c_0}\left(|\mathbf{x}|-\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)}}{|\mathbf{x}|^{\frac{d-1}{2}}}\left[i\frac{\omega}{c_0} - \frac{1}{|\mathbf{x}|}\left(\frac{\omega^2}{2c_0^2}\frac{|\mathbf{x}|^2|\mathbf{y}|^2-(\mathbf{x}\cdot\mathbf{y})^2}{|\mathbf{x}|^2} - \frac{d-1}{2}\frac{i\omega}{c_0}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} + \frac{d^2-1}{8}\right)\right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right). \tag{A.2}$$

*Proof.* Setting

$$\widetilde{K}(\mathbf{x}-\mathbf{y}) := \frac{1}{|\mathbf{x}-\mathbf{y}|^{\frac{d-2}{2}}} H^{(1)}_{\frac{d-2}{2}}\left(\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}|\right), \tag{A.3}$$

we have

$$\partial_{|\mathbf{x}|}\widetilde{K}(\mathbf{x}-\mathbf{y}) = \frac{\mathbf{x}}{|\mathbf{x}|}\cdot\nabla\widetilde{K}(\mathbf{x}-\mathbf{y}) = \frac{|\mathbf{x}|^2-\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\frac{1}{|\mathbf{x}-\mathbf{y}|^{\frac{d}{2}}}\left[\frac{\omega}{c_0}\left(H^{(1)}_{\frac{d-2}{2}}\right)'\left(\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}|\right) + \left(1-\frac{d}{2}\right)\frac{1}{|\mathbf{x}-\mathbf{y}|}H^{(1)}_{\frac{d-2}{2}}\left(\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}|\right)\right]. \tag{A.4}$$

Using the asymptotic behavior of $H_p^{(1)}$ for large arguments [28, Sect. 10.17(i–ii)]

$$H_p^{(1)}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\left(x-\frac{2p+1}{4}\pi\right)} \left(\frac{1}{x^{1/2}} + \frac{i\left(4p^2-1\right)}{8x^{3/2}}\right) + \mathcal{O}\left(\frac{1}{x^{5/2}}\right),$$

$$\frac{d}{dx}H_p^{(1)}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} e^{i\left(x-\frac{2p+1}{4}\pi\right)} \left(\frac{i}{x^{1/2}} - \frac{4p^2+3}{8x^{3/2}}\right) + \mathcal{O}\left(\frac{1}{x^{5/2}}\right), \quad x \gg 1,$$

we can write (A.3) and (A.4), respectively, as

$$\widetilde{K}(\mathbf{x}-\mathbf{y}) = \left(\frac{2c_0}{\pi\omega}\right)^{1/2} \frac{e^{-i(d-1)\pi/4}}{|\mathbf{x}-\mathbf{y}|^{\frac{d-1}{2}}} e^{i\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}|} \left[1 + \frac{ic_0}{8\omega}(d-3)(d-1)\frac{1}{|\mathbf{x}-\mathbf{y}|}\right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|^{\frac{d+3}{2}}}\right), \tag{A.5}$$

$$\partial_{|\mathbf{x}|}\widetilde{K}(\mathbf{x}-\mathbf{y}) = \left(\frac{2c_0}{\pi\omega}\right)^{1/2} \frac{|\mathbf{x}|^2 - \mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} \frac{e^{-i(d-1)\pi/4}}{|\mathbf{x}-\mathbf{y}|^{\frac{d+1}{2}}} e^{i\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}|} \times \left[i\frac{\omega}{c_0} + \left(1 - \frac{d}{2} - \frac{(d-2)^2+3}{8}\right)\frac{1}{|\mathbf{x}-\mathbf{y}|}\right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}-\mathbf{y}|^{\frac{d+5}{2}}}\right). \tag{A.6}$$

By using the identity

$$|\mathbf{x}-\mathbf{y}| = |\mathbf{x}|\left(1 - 2\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2}\right)^{1/2},$$

and the Taylor expansion of $(1+z)^{-\gamma/2}$ with $z := -2\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2}$ about $z = 0$, we obtain, for $|\mathbf{x}| \gg 1$,

$$\frac{1}{|\mathbf{x}-\mathbf{y}|^\gamma} = \frac{1}{|\mathbf{x}|^\gamma}\left(1 + \gamma\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}\right) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\gamma+2}}\right).$$

We will use this formula with $\gamma = \frac{d-1}{2}, \frac{d+1}{2}, \frac{d+3}{2}$. Moreover, using the Taylor expansions of $(1+z_1)^{1/2}$ and $\exp(z_2)$ with $z_1 := -2\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2}$ and $z_2 := i\frac{\omega}{c_0}|\mathbf{x}|\left[\left(1 - \frac{2\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2}\right)^{1/2} - 1 + \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}\right]$ about $z_1 = z_2 = 0$, we obtain

$$\begin{aligned}
e^{i\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}|} &= e^{i\frac{\omega}{c_0}\left(|\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)} e^{i\frac{\omega}{c_0}|\mathbf{x}|\left[\left(1-\frac{2\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}+\frac{|\mathbf{y}|^2}{|\mathbf{x}|^2}\right)^{1/2} - 1 + \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}\right]} \\
&= e^{i\frac{\omega}{c_0}\left(|\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)} \left(1 + i\frac{\omega}{c_0}|\mathbf{x}|\left[\left(1 - \frac{2\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2} + \frac{|\mathbf{y}|^2}{|\mathbf{x}|^2}\right)^{1/2} - 1 + \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2}\right]\right) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^2}\right) \\
&= e^{i\frac{\omega}{c_0}\left(|\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)} \left(1 + i\frac{\omega}{2c_0}\frac{|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2}{|\mathbf{x}|^3}\right) + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^2}\right).
\end{aligned}$$

Therefore, we get from (A.5) and (A.6)

$$\tilde{K}(\mathbf{x}-\mathbf{y}) = \left(\frac{2c_0}{\pi\omega}\right)^{1/2} \frac{e^{-i(d-1)\pi/4}}{|\mathbf{x}|^{\frac{d-1}{2}}} e^{i\frac{\omega}{c_0}\left(|\mathbf{x}|-\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)} \left[1 + \frac{1}{|\mathbf{x}|}\left(i\frac{c_0}{\omega}\frac{(d-3)(d-1)}{8} + \frac{d-1}{2}\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} + i\frac{\omega}{2c_0}\frac{|\mathbf{x}|^2|\mathbf{y}|^2-(\mathbf{x}\cdot\mathbf{y})^2}{|\mathbf{x}|^2}\right)\right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right), \tag{A.7}$$

$$\partial_{|\mathbf{x}|}\tilde{K}(\mathbf{x}-\mathbf{y}) = \left(\frac{2c_0}{\pi\omega}\right)^{1/2} \frac{e^{-i(d-1)\pi/4}}{|\mathbf{x}|^{\frac{d-1}{2}}} e^{i\frac{\omega}{c_0}\left(|\mathbf{x}|-\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|}\right)} \left[i\frac{\omega}{c_0} - \frac{1}{|\mathbf{x}|}\left(\frac{\omega^2}{2c_0^2}\left(|\mathbf{y}|^2 - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{|\mathbf{x}|^2}\right) + i\frac{\omega}{2c_0}(1-d)\frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} + \frac{d^2-1}{8}\right)\right] + \mathcal{O}\left(\frac{1}{|\mathbf{x}|^{\frac{d+3}{2}}}\right). \tag{A.8}$$

Since $K(\mathbf{x}) = \frac{i}{4}\left(\frac{\omega}{2\pi c_0}\right)^{\frac{d-2}{2}} \tilde{K}(\mathbf{x})$, estimates (A.7) and (A.8) imply (A.1) and (A.2). □

**Lemma A.2.** *Let $a, b > 0$, and define*

$$J := \int_0^1 \frac{dx}{(x^2+a^2)^{b/2}} \geq 0. \tag{A.9}$$

*Then, we have*

$$J \leq \begin{cases} C_{1,b}, & b<1, \\ \log\left(\frac{1}{a}+\sqrt{1+\frac{1}{a^2}}\right), & b=1, \\ C_{2,b}\frac{1}{a^{b-1}}, & b>1, \end{cases} \tag{A.10}$$

*where* $C_{1,b} := \frac{1}{1-b}$, $C_{2,b} := \int_0^\infty \frac{dx}{(1+x^2)^{b/2}}$.

*Proof.* After the change of variable $x \mapsto z := x/a$, we have

$$J = \frac{1}{a^{b-1}} \int_0^{1/a} \frac{dz}{(z^2+1)^{b/2}}.$$

Using

$$\int_0^{1/a} \frac{dz}{(z^2+1)^{b/2}} \leq \int_0^{1/a} \frac{dz}{z^b} = \frac{a^{b-1}}{1-b}$$

when $b < 1$,

$$\int_0^{1/a} \frac{dz}{(z^2+1)^{b/2}} \leq \int_0^{\infty} \frac{dz}{(z^2+1)^{b/2}} =: C_{2,b}$$

when $b > 1$, and

$$J = \int_0^{1/a} \frac{dz}{(z^2+1)^{1/2}} = \log\left(\frac{1}{a}+\sqrt{1+\frac{1}{a^2}}\right)$$

when $b = 1$, the estimate (A.10) follows immediately. □

**Lemma A.3.** *Let $a > 0$. For $t \gg 1$, we have the following estimate*

$$\int_0^a \frac{e^{-ixt}}{x^{1/2}} dx = \left(\frac{\pi}{2t}\right)^{1/2} (1-i) + \mathcal{O}\left(\frac{1}{t}\right). \tag{A.11}$$

*Proof.* Making a change of variable $x \mapsto z(x) := \sqrt{xt}$, we have

$$I(t) := \int_0^a \frac{e^{-ixt}}{x^{1/2}} dx = \frac{2}{\sqrt{t}} \int_0^{\sqrt{at}} e^{-iz^2} dz. \tag{A.12}$$

Since the integrand in (A.12) is analytic in $z$, we can invoke the Cauchy theorem to deform the integration contour in the complex plane. In particular, we choose the new contour $\Gamma_1 \cup \Gamma_2$ that consists of two parts: the straight line segment $\Gamma_1$ and the circular arc $\Gamma_2$. This contour is traversed counterclockwise with $\Gamma_1$ and $\Gamma_2$ defined, respectively, as

$$\Gamma_1 := \left\{ z \in \mathbb{C} : \, z = re^{-i\pi/4}, \, r \in (0, R) \right\},$$
$$\Gamma_2 := \left\{ z \in \mathbb{C} : \, z = Re^{i\phi}, \, \phi \in \left(-\frac{\pi}{4}, 0\right) \right\},$$

where for the sake of brevity, we have set $R := \sqrt{at}$. In other words, we can write

$$\begin{aligned} I(t) =& \frac{2}{\sqrt{t}} \left( \int_{\Gamma_1} e^{-iz^2} dz + \int_{\Gamma_2} e^{-iz^2} dz \right) \\ =& \frac{2(1-i)}{\sqrt{2t}} \int_0^R e^{-r^2} dr + \frac{2iR}{\sqrt{t}} \int_{-\pi/4}^0 \exp\left(-iR^2 e^{2i\phi} + i\phi\right) d\phi. \end{aligned} \tag{A.13}$$

Note that, for $R = \sqrt{at} \gg 1$, we have

$$\begin{aligned} \int_0^R e^{-r^2} dr &= \int_0^\infty e^{-r^2} dr - \int_R^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2} - e^{-R^2} \int_0^\infty e^{-r^2 - 2Rr} dr \\ &= \frac{\sqrt{\pi}}{2} + \mathcal{O}\left(e^{-at}\right), \end{aligned} \tag{A.14}$$

$$\begin{aligned} \int_{-\frac{\pi}{4}}^0 \exp\left(-iR^2 e^{2i\phi} + i\phi\right) d\phi =& \frac{1}{2R^2} \int_{-\frac{\pi}{4}}^0 \left[\frac{\partial}{\partial\phi} \exp\left(-iR^2 e^{2i\phi}\right)\right] e^{-i\phi} d\phi \\ =& \frac{1}{2R^2} \left( e^{-iR^2} - \frac{1+i}{\sqrt{2}} e^{-R^2} \right) \\ &+ \frac{i}{2R^2} \int_{-\frac{\pi}{4}}^0 \exp\left(-iR^2 e^{2i\phi} - i\phi\right) d\phi \\ =& \mathcal{O}\left(\frac{1}{R^2}\right) = \mathcal{O}\left(\frac{1}{t}\right). \end{aligned} \tag{A.15}$$

Inserting (A.14) and (A.15) into (A.13) furnishes the claimed estimate (A.11). ☐

**Lemma A.4.** *Let $F \in C^2(\mathbb{R}_+)$ satisfy for some $C_1 > 0$*

$$|F(t)| + |F''(t)| \le \frac{C_1}{(1+t^2)^{1/2}}, \quad t \ge 0. \tag{A.16}$$

*Then*

(A.17) $$|F'(t)| \leq \frac{C_2}{(1+t^2)^{1/2}}, \quad t \geq 0$$

*holds true with some* $C_2 > 0$.

*Proof.* For all $N \in \mathbb{N}_0$, we have

$$\|F\|_{L^\infty(N,N+1)} + \|F''\|_{L^\infty(N,N+1)} \leq \frac{C_1}{(1+N^2)^{1/2}}.$$

By interpolation (see [6, Prop. 2.2]), we deduce the same estimate also for $F'$, with some generic constant $\tilde{C}_1 > 0$ independent of $N$. Hence

$$|F'(t)| \leq \frac{\tilde{C}_1}{(1+\lfloor t \rfloor^2)^{1/2}}, \quad t \geq 0,$$

where $\lfloor t \rfloor$ is the floor function. Using

$$1 + \lfloor t \rfloor^2 = \frac{1+\lfloor t \rfloor^2}{1+t^2}\left(1+t^2\right) \geq \frac{1+(t-1)^2\chi_{(1,\infty)}(t)}{1+t^2}\left(1+t^2\right) \geq C_0\left(1+t^2\right),$$

with some constant $C_0 > 0$, we deduce (A.17) with $C_2 = \tilde{C}_1 / C_0^{1/2}$. □

## Appendix B.

We test our results numerically on an example where the material parameters $\alpha$, $\beta$ and the source term $F$ are radially symmetric. Namely, for $r := |\mathbf{x}|$, $\mathbf{x} \in \mathbb{R}^d$, we choose

(B.1) $$\alpha(r) := 2\chi_{[0,2)}(r) + \frac{1}{2}\chi_{[2,4)}(r)\left(3 + \cos\left(\frac{\pi}{2}(r-2)\right)\right) + \chi_{[4,\infty)}(r),$$

(B.2) $$\beta(r) := 1 + \chi_{(3,7)}(r)\left(1 + \cos\left(\frac{\pi}{2}(r-5)\right)\right),$$

(B.3) $$F(r) := 10\chi_{(0,8/3)}(r)\left(1 + \cos\left(\pi\left(\frac{3}{4}r - 1\right)\right)\right),$$

where $\chi$ denotes the characteristic function. Observe that, for this choice of $\alpha$ and $\beta$, the background medium parameters are $\alpha_0 = \beta_0 = 1$. We illustrate the functions in (B.1), (B.2), and (B.3) in Figure 1(a).

We fix the dimension $d \in \{1,2,3\}$ and the frequency $\omega = \pi/4$, and let $U(\mathbf{x})$ and $u(\mathbf{x},t)$ be the solutions to problems (1.1) and (1.2), respectively.

As in Table 1, we define

$$u^{\text{DIFF}}(\mathbf{x},t) := \begin{cases} u(\mathbf{x},t) - e^{-i\omega t}U(\mathbf{x}) & \text{if } d = 2,3, \\ u(\mathbf{x},t) - e^{-i\omega t}U(\mathbf{x}) - U_\infty & \text{if } d = 1, \end{cases}$$

where $U_\infty$ is the constant in (1.4).

We consider the bounded domain $B_{R_0} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| < R_0\}$ with $R_0 = 5$, and set

(B.4) $$\mathcal{E}(t) := \left(\|u^{\text{DIFF}}(\cdot,t)\|^2_{L^2(B_{R_0})} + \|\partial_t u^{\text{DIFF}}(\cdot,t)\|^2_{L^2(B_{R_0})}\right)^{1/2}.$$

Exploiting the radial symmetry, we rewrite problems (1.1) and (1.2) in the $(r,t)$-variables and we solve them numerically on the domain $(0,R)\times(0,T)$ with $R=120$ and $T=240$. For the time-dependent wave problem, we use finite differences in space on a uniform grid of size $6\cdot 10^{-2}$, and the Leapfrog method in time on a uniform grid of size $1.33\cdot 10^{-2}$. We solve the Helmholtz problem on the same spatial grid.

The following second-order radiation conditions have been used at $r=R$:

$$\partial_t u\,(R,t) = -\sqrt{\frac{\alpha_0}{\beta_0}}\left[\partial_r u\,(R,t) + \frac{1}{R}\frac{1-\delta_{d,1}}{1+\delta_{d,2}}u\,(R,t)\right], \quad t\geq 0, \tag{B.5}$$

$$U'\,(R) = \left[i\omega\sqrt{\frac{\beta_0}{\alpha_0}} - \frac{1}{R}\frac{1-\delta_{d,1}}{1+\delta_{d,2}}\right]U\,(R), \tag{B.6}$$

where $\delta$ denotes the Kronecker delta symbol (e.g., see [13, eq. (1.27)] and [19, eq. (7.10)] for $d=2$ and $d=3$, respectively). Note that these radiation conditions are exact in case $d=1$.

Central finite differences stencils were used to approximate first-order derivatives and ghost points were added at the boundaries $r=0$ and $r=R$. In doing so, the following relations were instrumental

$$\partial_t^2 u\,(0,t) - \frac{d\alpha\,(0)}{\beta\,(0)}\partial_r^2 u\,(0,t) = e^{-i\omega t}F\,(0)\,,$$

$$-\omega^2 U\,(0) - \frac{d\alpha\,(0)}{\beta\,(0)}U''\,(0) = F\,(0)\,.$$

These equations are obtained by passing to the limit $r\to 0$ in the equations in (1.1) and (1.2), and using the boundary conditions at $r=0$: $\partial_r u\,(0,t)=0$, $U'\,(0)=0$.

The quantity $\mathcal{E}\,(t)$ defined by (B.4) was computed from the numerical solution of (1.1) and (1.2), and is shown in Figure 1(b)–1(d). In particular, in Figure 1(b), we observe a much faster decay in time for $d=1$ and $d=3$ than for $d=2$. The semilogarithmic plot in Figure 1(c) shows this decay to be exponential (up to the saturation due to numerical errors for small quantities at large times). A linear region for large times is observed in logarithmic plot in Figure 1(d) for $d=2$. As a comparison, we have plotted in black a line of slope $-1$. This clearly illustrates an algebraic convergence, corroborating the sharpness of the decay estimate in Theorem 1.4 for this case.

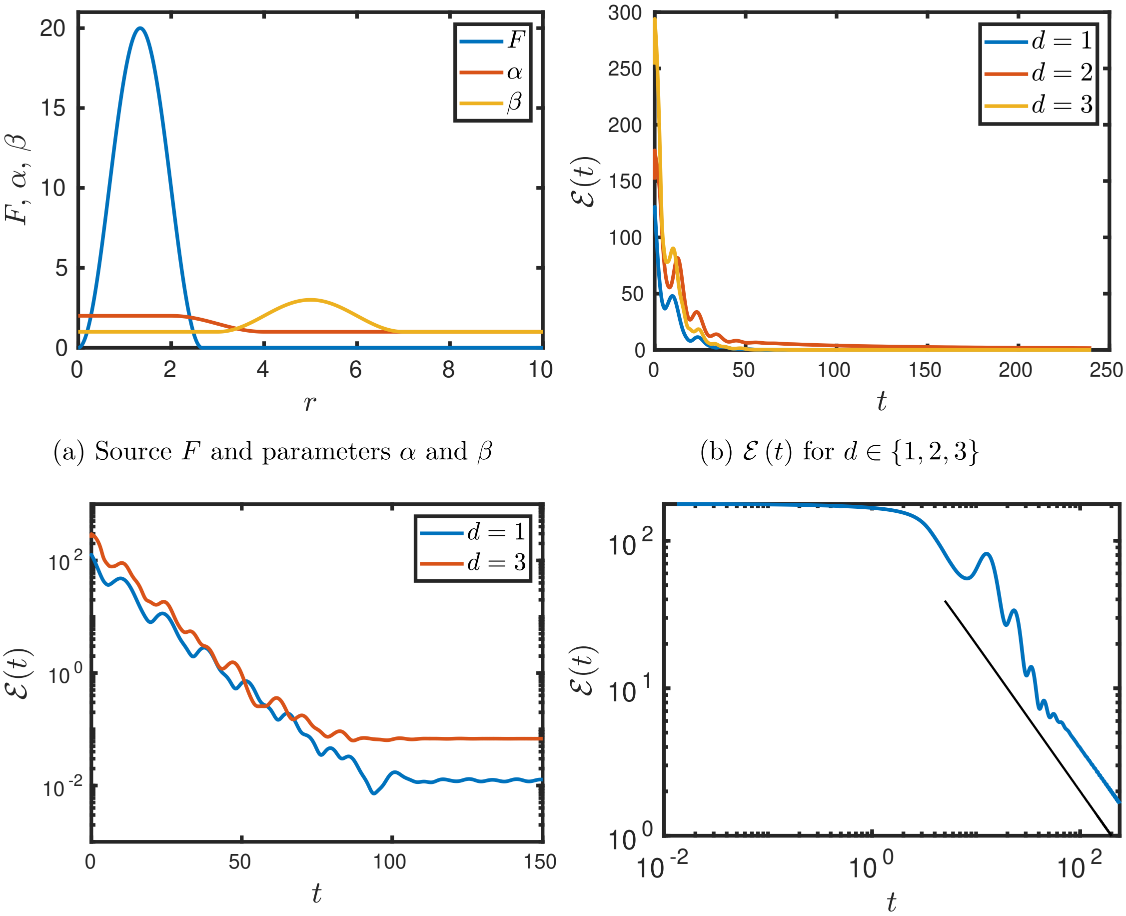
## References

[1] D. Appelo, F. Garcia, and O. Runborg. Waveholtz: Iterative solution of the Helmholtz equation via the wave equation. *SIAM Journal on Scientific Computing*, 42(4):A1950–A1983, 2020.

[2] A. Arnold, S. Geevers, I. Perugia, and D. Ponomarev. An adaptive finite element method for high-frequency scattering problems with smoothly varying coefficients. *Computers & Mathematics with Applications*, 109:1–14, 2022.

[3] A. Arnold, S. Geevers, I. Perugia, and D. Ponomarev. On the exponential time-decay for the one-dimensional wave equation with variable coefficients. *Communications on Pure and Applied Analysis*, 21(10):3389–3405, 2022.

[4] M. Beals. Global time decay of the amplitude of a reflected wave. *Partial Differential Equations and Mathematical Physics. Progress in Nonlinear Differential Equations and Their Applications*, 21:25–44, 1996.




(a) Source $F$ and parameters $\alpha$ and $\beta$

(b) $\mathcal{E}(t)$ for $d \in \{1, 2, 3\}$

(c) $\mathcal{E}(t)$ for $d \in \{1, 3\}$ in semilog scale

(d) $\mathcal{E}(t)$ for $d = 2$ in log scale

FIGURE 1. Large-time convergence for the radially symmetric example with the data as in (B.1)–(B.3).


[5] J.-M. Bouclet and N. Burq. Sharp resolvent and time-decay estimates for dispersive equations on asymptotically Euclidean backgrounds. *Duke Mathematical Journal*, 170(11):2575 – 2629, 2021.

[6] H. Brezis and P. Mironescu. Gagliardo-Nirenberg inequalities and non-inequalities: the full story. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 35(5):1355–1376, 2018.

[7] M.-O. Bristeau, R. Glowinski, and J. Périaux. Controllability methods for the computation of time-periodic solutions; application to scattering. *Journal of Computational Physics*, 147(2):265–292, 1998.

[8] C. Carvalho, P. Ciarlet Jr., and C. Scheid. Limiting amplitude principle and resonances in plasmonic structures with corners: numerical investigation. *Computer Methods in Applied Mechanics and Engineering*, 388:114207, 2022.

[9] M. Cassier, C. Hazard, and P. Joly. Spectral theory for Maxwell's equations at the interface of a metamaterial. Part I: Generalized Fourier transform. *Communications in Partial Differential Equations*, 42(11):1707–1748, 2017.

[10] M. Cassier, C. Hazard, and P. Joly. Spectral theory for Maxwell's equations at the interface of a metamaterial. Part II: Limiting absorption, limiting amplitude principles and interface resonance. *Communications in Partial Differential Equations*, 47(6):1217–1295, 2022.

[11] D. Eidus. On the principle of limiting absorption [translation from Russian]. *American Mathematical Society Translations*, 47(2):157–191, 1965.

[12] D. Eidus. The principle of limit amplitude. *Russian Mathematical Surveys*, 24:97–167, 1969.

[13] B. Engquist and A. Majda. Absorbing boundary conditions for numerical simulation of waves. *Proceedings of the National Academy of Sciences*, 74(5):1765–1766, 1977.

[14] B. Engquist and H. Zhao. Approximate separability of the Green's function of the Helmholtz equation in the high frequency limit. *Communications on Pure and Applied Mathematics*, 71(11):2220–2274, 2018.

[15] L. Evans. *Partial Differential Equations (2nd edition)*. AMS, 2010.

[16] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer, 1977.

[17] R. Glowinski and T. Rossi. A mixed formulation and exact controllability approach for the computation of the periodic solutions of the scalar wave equation. I. Controllability problem formulation and related iterative solution. *Comptes Rendus Mathématique. Académie des Sciences. Paris*, 343(7):493–498, 2006.

[18] I. G. Graham, O. R. Pembery, and E. A. Spence. The Helmholtz equation in heterogeneous media: a priori bounds, well-posedness, and resonances. *Journal of Differential Equations*, 266(6):2869–2923, 2019.

[19] M. J. Grote and J. B. Keller. Exact Nonreflecting Boundary Conditions for the Time Dependent Wave Equation. *SIAM Journal on Applied Mathematics*, 55(2):280–297, 1995.

[20] M. J. Grote, F. Nataf, J. H. Tang, and P.-H. Tournier. Parallel controllability methods for the Helmholtz equation. *Computer Methods in Applied Mechanics and Engineering*, 362:112846, 2020.

[21] M. J. Grote and J. H. Tang. On controllability methods for the Helmholtz equation. *Journal of Computational and Applied Mathematics*, 358:306–326, 2019.

[22] E. Heikkola, S. Mönkölä, A. Pennanen, and T. Rossi. Controllability method for the Helmholtz equation with higher-order discretizations. *Journal of Computational Physics*, 225(2):1553–1576, 2007.

[23] N. Iwasaki. On the principle of limiting amplitude. *Publications of RIMS Kyoto University - Series A*, 3:373–392, 1968.

[24] O. Ladyzhenskaya. On the principle of limit amplitude. *Uspekhi Mat. Nauk*, 12(3):161–164, 1957.

[25] C. Morawetz. The limiting amplitude principle. *Communications on Pure and Applied Mathematics*, 15(3):349–361, 1962.

[26] F. Odeh. Principles of Limiting Absorption and Limiting Amplitude in Scattering Theory. I. Schrödinger's Equation. *Journal of Mathematical Physics*, 2(6):794–800, 1961.

[27] F. Odeh. Principles of Limiting Absorption and Limiting Amplitude in Scattering Theory. II. The Wave Equation in an Inhomogeneous Medium. *Journal of Mathematical Physics*, 2(6):800–802, 1961.

[28] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions*. Cambridge University Press, 2010.

[29] A. Ramm. On the limiting amplitude principle. *Diff. Uravn.*, 4(4):714–720, 1968.

[30] A. Ramm. Necessary and suffient conditions of validity of the limiting amplitude principle. *Izv. Vysch. Uch. Zaved.*, 5(1):96–102, 1978.

[31] J. Shapiro. Local energy decay for Lipschitz wavespeeds. *Communications in Partial Differential Equations*, 43(5):839–858, 2018.

[32] C. C. Stolk. A time-domain preconditioner for the Helmholtz equation. *SIAM Journal on Scientific Computing*, 43(6), 2021.

[33] H. Tamura. Resolvent estimates at low frequencies and limiting amplitude principle for acoustic propagators. *Journal of Mathematical Society of Japan*, 41(4):549–575, 1989.

[34] A. Tikhonov and A. Samarsky. On the radiation principle. *ZHETF*, 18(2):97–108, 1948.

[35] A. Tikhonov and A. Samarsky. *Equations of Mathematical Physics*. Pergamon Press, 1963.

[36] B. Vainberg. Principles of radiation, limit absorption and limit amplitude in the general theory of partial differential equations. *Russian Mathemathical Surveys*, 21(3):115–193, 1966.