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Justification of a nonlinear Schrödinger model for laser beams in photopolymers

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Abstract. A nonstationary model that relies on the spatial nonlinear Schrödinger (NLS) equation with the time-dependent refractive index describes laser beams in photopolymers. We consider a toy problem, when the rate of change of refractive index is proportional to the squared amplitude of the electric field and the spatial domain is a plane. After formal derivation of the NLS approximation from a two-dimensional quasilinear wave equation, we establish local well-posedness of the original and reduced models and perform rigorous justification analysis to control smallness of the approximation error for appropriately small times.

Mathematics Subject Classification. 35L52 · 35Q55 · 35Q60 · 35Q61.

Keywords. Nonlinear Schrödinger equation · Justification of amplitude equations · A priori energy estimates · Local well-posedness.

1. Introduction

Mathematical models for laser beams in photochemical materials used in the physical literature [14] are based on a spatial nonlinear Schrödinger (NLS) equation with a time-dependent refractive index. These models are normally derived from Maxwell equations using heuristic arguments and qualitative approximations (see e.g., [15,19]). Numerical simulations of such models are performed by experimentalists [9,21] for theoretical explanations of complicated dynamics of laser beams in photopolymers. The complexity of the NLS equation modeling photochemical materials is related to the fact that the spatial coordinate in the direction of the beam propagation serves as an evolution time in the NLS equation, whereas the nonlinear refractive index depends slowly on the temporal coordinate. Physically, the NLS approximation describes laser beams in emerging waveguides in polymers which affect shape and dynamics of the pulse via nonlinear refractive index.

In the present work, we study how to justify a time-dependent NLS model from a toy model resembling the Maxwell equations. The toy model is written as a system of a two-dimensional quasilinear wave equation and an empirical relation for the change of the refractive index. Although the justification procedure for the classical nonlinear Schrödinger equation is well-known [8], we emphasize that no results are available in the mathematical literature on the justification of the spatial NLS equation with a time-dependent refractive index. Among other relevant results, we mention the work of J. Rauch and his collaborators on dispersive nonlinear geometric optics in the context of Maxwell–Bloch equations [3–7] and the works of G. Schneider and his collaborators on justification of various versions of the NLS equation in the context of Maxwell equations [11,12,17,18].

1.1. Toy model

A photopolymer occupies typically a half space $z \geq 0$, and its face $z = 0$ is exposed to a laser beam. If the beam is localized in the x-direction and uniform in the y-direction, then the electric field has polarization in the y-direction with the amplitude E being y-independent; hence, $\mathbf{E}(x,z,t) = (0, E(x,z,t), 0)$

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is the electric field. The initial beam is assumed to be spatially wide-spreaded, small in amplitude, and monochromatic in time.

Neglecting polarization effects and uniform material losses, the electric field satisfies a two-dimensional quasilinear wave equation in the form

$$\partial_x^2 E + \partial_z^2 E - n^2 \partial_t^2 E = 0, \tag{1.1}$$

where n is referred to as the refractive index of the photopolymer. The refractive index n changes in time t because of the nonlinear effects induced by the squared amplitude of the electric field E.

Let us write the squared refractive index in the form $n^2 = 1 + m$ and assume that the rate of change of m is governed by the empirical relation

$$\partial_t m = E^2. \tag{1.2}$$

We note that all physical constants in the system (1.1)–(1.2) are normalized to unity.

The system (1.1)–(1.2) approximates a more complicated system of governing equations in the physical literature [14], for which the quasilinear wave equation is written via a polarization term

$$\partial_x^2 E + \partial_z^2 E - \partial_t^2 P = 0, \quad P = n^2 E, \tag{1.3}$$

whereas the correction of the refractive index $m = n^2 - 1$ is modeled by

$$\partial_t m = E^2 \left(1 - \frac{m}{m_s}\right) \quad \Rightarrow \quad m = m_s \left(1 - e^{-\frac{1}{m_s}\int_0^t E^2 \mathrm{d}t'}\right), \tag{1.4}$$

where m_s is the constant level of saturation for m. Justification of the system (1.3)–(1.4) is expected to be analogous to the results which are presented here for the simplified system (1.1)–(1.2).

It is worth mentioning that the realistic three-dimensional problems for laser beams in photochemical materials can also be treated with similar analysis. Even though the starting system will then take a form of the system of coupled Maxwell equations, functional embedding of $H^2(\mathbb{R}^3)$ to $C_b(\mathbb{R}^3)$ allows us to close the estimates within the same energy levels as those in the considered two-dimensional case.

1.2. Asymptotic balance

Let us seek for the asymptotic solution to the system (1.1)–(1.2) by using the multi-scale expansion [16,19]

$$E\,(x,z,t) = \epsilon^p A(X,Z,T) e^{i\omega_0(z-t)} + \text{c.c.}, \quad m\,(x,z,t) = \epsilon^r M(X,Z,T), \tag{1.5}$$

where c.c. stands for complex conjugated terms, $X = \epsilon x$, $Z = \epsilon^q z$, $T = \epsilon^s t$ are slow variables, and p, q, s, $r > 0$ are exponents to be specified.

We want to choose the exponents p, q, s, and r such that A is governed by the NLS equation, which has first-order partial derivatives of A in Z, second derivative in X, and a nonlinear term proportional to $m_0 A$ at the leading order of ϵ (that is $\mathcal{O}\left(\epsilon^{p+2}\right)$ due to the term $\partial_x^2 E$). At the same time, Eq. (1.2) must enforce the rate of change of M in T to be of order $\mathcal{O}\,(1)$ at the leading order of ϵ (that is $\mathcal{O}\left(\epsilon^{2p}\right)$ due to the term E^2). These requirements lead to the choice

$$q = 2, \quad r = 2, \quad s = 2p - 2, \tag{1.6}$$

which still leaves parameter p to be defined.

To show (1.6), we substitute (1.5) in (1.1) and (1.2) to obtain:

$$\epsilon^p \left[\epsilon^2 \partial_X^2 A + 2i\omega_0 \left(\epsilon^q \partial_Z A + \epsilon^s \partial_T A\right) + \epsilon^r \omega_0^2 M A\right] e^{i\omega_0(z-t)} + \text{c.c.} + \text{h.o.t.} = 0, \tag{1.7}$$

and

$$\epsilon^{r+s} \partial_T M = \epsilon^{2p} \left|A\right|^2 + \left(\epsilon^{2p} A^2 e^{2i\omega_0(z-t)} + \text{c.c.}\right) + \text{h.o.t.}, \tag{1.8}$$

where h.o.t. stands for the higher order terms.

From Eq. (1.7), the balance occurs for $q = 2$, $r = 2$, and $s \geq 2$. From Eq. (1.8), the balance occurs for $r + s = 2p$, hence $s = 2p - 2$, and the balance (1.6) is justified. Note that, the second term in Eq. (1.8) induces the second harmonic, which will be further included in the equation for a residual term.

If $s = 2$, the leading-order terms of the system (1.7)–(1.8) read as follows:

$$\partial_X^2 A + 2i\omega_0 \left(\partial_Z A + \partial_T A\right) + \omega_0^2 M A = 0 \tag{1.9}$$

and

$$\partial_T M = 2 \left|A\right|^2, \tag{1.10}$$

which will be the subject of our studies.

If $s > 2$, the leading-order terms of Eq. (1.7) yield the spatial NLS equation

$$\partial_X^2 A + 2i\omega_0 \partial_Z A + \omega_0^2 M A = 0. \tag{1.11}$$

Because M depends on T by means of the same Eq. (1.10), A depends on T implicitly in Eq. (1.11). The system (1.10)–(1.11) was used in the previous works on photochemical materials (see review in [14]). Unfortunately, our method does not allow us to justify the system (1.10)–(1.11) at the present time. For example, the energy (L^2-norm) of the solution A on the plane (X, Z) is infinite because the integral of $|A|^2$ in X is independent of the Z-variable.

Our task is to justify the system (1.9)–(1.10), where the time evolution of A is uniquely determined. To avoid nonvanishing boundary terms arising in energy method when integrating by parts, we shall consider solutions of the original system (1.1)–(1.2) on the whole plane $(x, z) \in \mathbb{R}^2$ supplemented by the initial conditions at $t = 0$.

To summarize, in the case $s = 2$, we choose the scaling $X = \epsilon x$, $Z = \epsilon^2 z$, $T = \epsilon^2 t$ and represent exact solution to the system (1.1)–(1.2) as

$$E(x, z, t) = \epsilon^2 \left(A(X, Z, T) e^{i\omega_0(z-t)} + \text{c.c.}\right) + U(x, z, t) \tag{1.12}$$

and

$$m(x, z, t) = \epsilon^2 M(X, Z, T) + N(x, z, t), \tag{1.13}$$

where $U(x, z, t)$ and $N(x, z, t)$ are error terms to estimate.

Let us denote

$$(R)_{n\omega_0} := \text{R}e^{in\omega_0(z-t)} + \text{c.c.}$$

where the complex envelope R at the nth harmonic may depend on X, Z, and T.

Feeding (1.12)–(1.13) into (1.1)–(1.2) and assuming validity of (1.9)–(1.10), we arrive at the system

$$\partial_x^2 U + \partial_z^2 U - \left(1 + \epsilon^2 M + N\right) \partial_t^2 U = -\epsilon^2 \left(R_2^{(U)}\right)_{\omega_0} N - \epsilon^6 \left(R_6^{(U)}\right)_{\omega_0} \tag{1.14}$$

and

$$\partial_t N = \epsilon^4 \left(A^2\right)_{2\omega_0} + 2\epsilon^2 \left(A\right)_{\omega_0} U + U^2, \tag{1.15}$$

where

$$\begin{aligned} R_2^{(U)} &= \omega_0^2 A + 2i\omega_0 \epsilon^2 \partial_T A - \epsilon^4 \partial_T^2 A, \\ R_6^{(U)} &= \partial_Z^2 A - \left(1 + \epsilon^2 M\right) \partial_T^2 A + 2i\omega_0 M \partial_T A. \end{aligned}$$

We shall now consider an initial-value problem, for which we formulate the main justification result.

1.3. Main result

For the system (1.1)–(1.2), we impose the following initial conditions

$$E|_{t=0} = \epsilon^2 A_0\left(\epsilon x, \epsilon^2 z\right) e^{i\omega_0 z} + \text{c.c.} =: E_0, \tag{1.16}$$

$$\partial_t E|_{t=0} = -i\omega_0 \epsilon^2 A_0\left(\epsilon x, \epsilon^2 z\right) e^{i\omega_0 z} + \epsilon^4 \partial_T A_0\left(\epsilon x, \epsilon^2 z\right) e^{i\omega_0 z} + \text{c.c.} =: E_1, \tag{1.17}$$

$$m|_{t=0} = 0, \tag{1.18}$$

where A_0 is the initial distribution of the beam for the nonlinear Schrödinger equation (1.9) and $\partial_T A_0$ is expressed explicitly from (1.9). Such initial conditions imply that the electrical field E at $t=0$ is already penetrated in the photopolymer but has not yet induced the change in the refractive index m. Note also that the conditions (1.16)–(1.18) imply that $U|_{t=0} = \partial_t U|_{t=0} = N|_{t=0} = 0$ in the system (1.14)–(1.15) for the error terms.

Our main result is the following justification theorem.

Theorem 1. *Given initial data $A_0 \in H^8\left(\mathbb{R}^2\right)$, let A, M be local solutions to the system (1.9)–(1.10) for $T \in [0, T_\infty)$ where $T_\infty > 0$ is the maximal existence time. There exist $\epsilon_0 > 0$ and $T_0 \in (0, T_\infty)$ such that for every $\epsilon \in (0, \epsilon_0)$ there is a unique solution E, m of the system (1.1)–(1.2) for $t \in \left[0, T_0/\epsilon^2\right]$ satisfying the initial conditions (1.16)–(1.18) and the bounds*

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|E - \epsilon^2 (A)_{\omega_0}\right\|_{H^3(\mathbb{R}^2)} = \mathcal{O}\left(\epsilon^{5/2}\right), \quad \sup_{t\in[0,T_0/\epsilon^2]} \left\|m - \epsilon^2 M\right\|_{H^2(\mathbb{R}^2)} = \mathcal{O}\left(\epsilon^{5/2}\right). \tag{1.19}$$

The methods of the proof of Theorem 1 are standard, e.g., energy methods and Gronwall inequality. The main technical difficulty arises because Eq. (1.1) is quasilinear and lacks the linear dispersion of the Klein–Gordon equation. As a result, estimates for the L^2-norm of the error terms can be obtained only by integration of the time derivative term and hence are $\mathcal{O}(\epsilon^2)$-larger compared to the L^2-norm of the spatial gradients and time derivatives of the error terms.

We note that the leading-order terms in the decompositions (1.12)–(1.13) are bounded by

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|\epsilon^2 (A)_{\omega_0}\right\|_{L^2} \leq 2\epsilon^2 \sup_{T\in[0,T_0]} \left(\int_{\mathbb{R}^2} \left|A\left(\epsilon x, \epsilon^2 z, T\right)\right|^2 \mathrm{d}x\mathrm{d}z\right)^{1/2} = \mathcal{O}(\epsilon^{1/2}) \tag{1.20}$$

and

$$\sup_{t\in[0,T_0/\epsilon^2]} \left\|\epsilon^2 M\right\|_{L^2} = \epsilon^2 \sup_{T\in[0,T_0]} \left(\int_{\mathbb{R}^2} \left|M\left(\epsilon x, \epsilon^2 z, T\right)\right|^2 \mathrm{d}x\mathrm{d}z\right)^{1/2} = \mathcal{O}(\epsilon^{1/2}), \tag{1.21}$$

with similar bounds for the L^2-norms of the derivatives. Therefore, the error terms in the decompositions (1.12)–(1.13) are $\mathcal{O}(\epsilon^2)$-smaller than the leading-order terms in the corresponding Sobolev norms. Also note that the L^2-norms of the leading-order terms are $\mathcal{O}(\epsilon^{3/2})$-larger compared with the order of their L^∞-norm because of integration in the slow variables $X = \epsilon x$ and $Z = \epsilon^2 z$.

We shall outline the proof of Theorem 1 in more detail. First, in Sect. 2, we review local existence of solutions of the system (1.1)–(1.2) in Sobolev spaces. By using Kato's theory [10], we prove existence of local solutions

$$E \in C\left([0,t_0], H^4\left(\mathbb{R}^2\right)\right) \cap C^1\left([0,t_0], H^3\left(\mathbb{R}^2\right)\right) \cap C^2\left([0,t_0], H^2\left(\mathbb{R}^2\right)\right), \tag{1.22}$$

for some $t_0 > 0$ which can be continued to $t_0' > t_0$ as long as

$$\sup_{t\in[0,t_0']} \left(\|E\|_{L^\infty} + \|\partial_t E\|_{L^\infty} + \|\nabla E\|_{L^\infty}\right) < \infty. \tag{1.23}$$

Solutions of the system are extended on the time intervals $[0, T_0/\epsilon^2]$ with an ϵ-independent T_0, for which local solutions of the NLS system (1.9)–(1.10) can be considered for any $A_0 \in H^s(\mathbb{R}^2)$ with $s > 1$. The functional analysis tools needed for our work are collected together in Appendix A.

The goal of Sect. 3 is to obtain sufficient estimates for the error terms U, N governed by (1.14)–(1.15) and hence to justify the approximation of solutions of the system (1.1)–(1.2) by solutions of the NLS system (1.9)–(1.10).

First, we execute near-identity transformations to move the residual terms of the system (1.14)–(1.15) to the $\mathcal{O}(\epsilon^8)$ order in the L^∞-norm or to the $\mathcal{O}(\epsilon^{13/2})$ order in the L^2-norm Then, using *a priori* energy estimates and Gronwall inequality, we bound terms such as $\|\partial_t U\|_{L^2}$ and $\|\nabla U\|_{L^2}$ on the time intervals $[0, T_0/\epsilon^2]$ by the $\mathcal{O}(\epsilon^{9/2})$ error of the inhomogeneous (source) terms in the L^2-norm. Because of the lack of the linear dispersion term, we use bounds like

$$\|U\|_{L^2} \leq \frac{T_0}{\epsilon^2} \sup_{t\in[0,T_0/\epsilon^2]} \|U_t\|_{L^2}\,, \tag{1.24}$$

which results in a larger $\mathcal{O}(\epsilon^{5/2})$ error in Theorem 1. Nevertheless, because the residual terms have been moved to the higher order by means of near-identity transformations, we are able to close the estimates for small values of T_0 that satisfy some technical constraints. This construction allows us to continue solutions to the time interval $[0, T_0/\epsilon^2]$ and to bound the error terms on these time intervals.

2. Local well-posedness theory

Before we proceed with the justification analysis, let us consider the question of local well-posedness of the wave system (1.1)–(1.2) and formulate a regularity criterion for the continuations of local solutions. We also obtain local well-posedness of the NLS system (1.9)–(1.10). Note that, many useful results from functional analysis are reviewed in Appendix A and labeled by the capital letters (instead of numbers).

2.1. Local well-posedness of the quasilinear wave system

Consider the quasilinear wave system

$$\begin{cases} \partial_x^2 E + \partial_z^2 E - (1+m)\,\partial_t^2 E = 0, \\ \partial_t m = E^2, \end{cases} \qquad (x,z)\in\mathbb{R}^2, \quad t\in\mathbb{R}_+, \tag{2.1}$$

subject to the initial conditions $m|_{t=0} = 0$, $E|_{t=0} = E_0$, and $E_t|_{t=0} = E_1$ for given E_0, $E_1 \in H^s\left(\mathbb{R}^2\right)$ with some $s \geq 0$, where H^s is the L^2-based Sobolev space. We can apply the local well-posedness theory for quasilinear symmetric hyperbolic systems [10,13,20] once we bring the quasilinear wave system (2.1) into a form of the first-order system associated with a symmetric matrix.

To symmetrize the system, we set

$$\mathbf{v} := \left(\partial_t E,\, \frac{\partial_x E}{(1+m)^{1/2}},\, \frac{\partial_z E}{(1+m)^{1/2}},\, E,\, \partial_x m,\, \partial_z m,\, m\right)^T. \tag{2.2}$$

Then, the wave system (2.1) is equivalent to the symmetric quasilinear first-order system

$$\partial_t \mathbf{v} + A_1(\mathbf{v})\,\partial_x \mathbf{v} + A_2(\mathbf{v})\,\partial_z \mathbf{v} = \mathbf{f}(\mathbf{v})\,, \tag{2.3}$$

where A_1, A_2 are matrices with the only nonzero elements

$$-\frac{1}{(1+v_7)^{1/2}} = -\frac{1}{(1+m)^{1/2}}$$

located at $(1,2)$–$(2,1)$ and $(1,3)$–$(3,1)$ entries, respectively, whereas $\mathbf{f}(\mathbf{v})$ is a vector field given by

$$\mathbf{f}(\mathbf{v}) = \left(\frac{v_2 v_5 + v_3 v_6}{2(1+v_7)^{3/2}}, -\frac{v_4^2 v_2}{2(1+v_7)}, -\frac{v_4^2 v_3}{2(1+v_7)}, v_1, 2(1+v_7)^{1/2} v_2 v_4, 2(1+v_7)^{1/2} v_3 v_4, v_4^2 \right)^T$$
$$= \left(\frac{\partial_x E \partial_x m + \partial_z E \partial_z m}{2(1+m)^2}, -\frac{E^2 \partial_x E}{2(1+m)^{3/2}}, -\frac{E^2 \partial_z E}{2(1+m)^{3/2}}, \partial_t E, 2E\partial_x E, 2E\partial_z E, E^2 \right)^T.$$

Note that, A_1, A_2, $\mathbf{f}$ have no explicit dependence on x, z and t.

The initial data for (2.3) are given by

$$\mathbf{v}|_{t=0} = (E_1, \partial_x E_0, \partial_z E_0, E_0, 0, 0, 0)^T. \tag{2.4}$$

By the Kato theory (see Theorems I–II in [10]), for any $\mathbf{v}_0 \in H^s(\mathbb{R}^2)$ with a fixed $s > \frac{5}{2}$, the Cauchy problem (2.3)–(2.4) admits unique local solution in class of functions

$$\mathbf{v} \in C\left([0,t_0], H^s(\mathbb{R}^2)\right) \cap C^1\left([0,t_0], H^{s-1}(\mathbb{R}^2)\right)$$

for some $t_0 > 0$. Moreover, the solution $\mathbf{v}$ depends on the initial data $\mathbf{v}_0$ continuously (Theorem III in [10]). We transfer this result in the following lemma, where we prefer to work with integer values of $s \geq 3$.

Lemma 1. *For any integer $s \geq 3$, there exists a unique local solution of the system (2.1) in the class of functions*

$$E \in C\left([0,t_0], H^{s+1}(\mathbb{R}^2)\right) \cap C^1\left([0,t_0], H^s(\mathbb{R}^2)\right) \cap C^2\left([0,t_0], H^{s-1}(\mathbb{R}^2)\right), \tag{2.5}$$
$$m \in C^1\left([0,t_0], H^{s+1}(\mathbb{R}^2)\right) \cap C^2\left([0,t_0], H^s(\mathbb{R}^2)\right) \cap C^3\left([0,t_0], H^{s-1}(\mathbb{R}^2)\right). \tag{2.6}$$

Moreover, the solution depends continuously on the initial data $E_0 \in H^{s+1}(\mathbb{R}^2)$ and $E_1 \in H^s(\mathbb{R}^2)$.

Proof. From the first and the last four entries in (2.2), we infer that, for any integer $s \geq 3$,

$$E \in C^1\left([0,t_0], H^s(\mathbb{R}^2)\right) \cap C^2\left([0,t_0], H^{s-1}(\mathbb{R}^2)\right), \tag{2.7}$$
$$m \in C\left([0,t_0], H^{s+1}(\mathbb{R}^2)\right) \cap C^1\left([0,t_0], H^s(\mathbb{R}^2)\right). \tag{2.8}$$

We shall now use the second and third entries in (2.2), which tell us that

$$J := \int_{\mathbb{R}^2} \left(\left[\partial_x^s \left(\frac{\partial_x E}{(1+m)^{1/2}} \right) \right]^2 + \left[\partial_z^s \left(\frac{\partial_z E}{(1+m)^{1/2}} \right) \right]^2 \right) \mathrm{d}x\mathrm{d}z$$

is a bounded continuous function of t on $[0,t_0]$. Without loss of generality, let us keep track of only x-derivatives.

By the Leibnitz differentiation rule, we have

$$\left[\partial_x^s \left(\frac{\partial_x E}{(1+m)^{1/2}} \right) \right]^2 = \left[\sum_{k=0}^{s} \binom{s}{k} \partial_x^{k+1} E \partial_x^{s-k} (1+m)^{-1/2} \right]^2$$
$$= \left[(1+m)^{-1/2} \partial_x^{s+1} E + \sum_{k=0}^{s-1} \binom{s}{k} \partial_x^{k+1} E \partial_x^{s-k} (1+m)^{-1/2} \right]^2,$$

where $\binom{s}{k}$ is a binomial coefficient. Denoting

$$\lambda := \left(\int_{\mathbb{R}^2} \frac{\left[\partial_x^{s+1} E\right]^2}{1+m} \mathrm{d}x\mathrm{d}z \right)^{1/2}, \quad \mu := \left(\int_{\mathbb{R}^2} \left[\sum_{k=0}^{s-1} \binom{s}{k} \partial_x^{k+1} E \partial_x^{s-k} (1+m)^{-1/2} \right]^2 \mathrm{d}x\mathrm{d}z \right)^{1/2},$$

we use the Cauchy–Schwarz inequality to estimate

$$\begin{aligned}\lambda^2 &\leq J - 2\int_{\mathbb{R}^2} \frac{\partial_x^{s+1}E}{(1+m)^{1/2}} \sum_{k=0}^{s-1} \binom{s}{k} \partial_x^{k+1}E\partial_x^{s-k}(1+m)^{-1/2}\,\mathrm{d}x\mathrm{d}z \\ &\quad - \int_{\mathbb{R}^2} \left[\sum_{k=0}^{s-1} \binom{s}{k} \partial_x^{k+1}E\partial_x^{s-k}(1+m)^{-1/2}\right]^2 \mathrm{d}x\mathrm{d}z \\ &\leq J + 2\lambda\mu.\end{aligned}$$

But then

$$\lambda^2 - 2\mu\lambda - J \leq 0 \quad \Rightarrow \quad \lambda \leq \mu + \sqrt{\mu^2 + J}.$$

Let us now show that $\mu < \infty$ for any $t \in [0, t_0]$. By the triangle inequality for L^2-norm, for some constant $C > 0$, we have

$$\begin{aligned}\mu &\leq C \sum_{k=0}^{s-1} \left(\int_{\mathbb{R}^2} \left(\partial_x^{k+1}E\right)^2 \left[\partial_x^{s-k}(1+m)^{-1/2}\right]^2 \mathrm{d}x\mathrm{d}z \right)^{1/2} \\ &\leq C \left[\|\partial_x E\|_{L^\infty} \left\|\partial_x^s (1+m)^{-1/2}\right\|_{L^2} + \sum_{k=1}^{s-1} \left\|\partial_x^{k+1}E\right\|_{L^2} \left\|\partial_x^{s-k}(1+m)^{-1/2}\right\|_{L^\infty} \right].\end{aligned}$$

The right-hand side of the last inequality is bounded for any $t \in [0, t_0]$ because $\|\partial_x^s E\|_{L^2}$, $\left\|\partial_x^{s+1} m\right\|_{L^2}$, $\|\partial_x E\|_{L^\infty}$, $\left\|\partial_x^{s-1} m\right\|_{L^\infty}$, $\|m\|_{L^\infty}$ are all bounded due to (2.7)–(2.8), as well as by Sobolev's embeddings (Proposition B) and Banach algebra of the L^∞-norm. Since $\mu < \infty$, then $\lambda < \infty$ holds for all $t \in [0, t_0]$.

Now, since $m|_{t=0} = 0$ and $\partial_t m = E^2 \geq 0$, we have $m(x, z, t) \geq 0$ for all $(x, z) \in \mathbb{R}^2$ and $t \in [0, t_0]$. Therefore, we obtain

$$\frac{1}{1 + \|m\|_{L^\infty}} \int_{\mathbb{R}^2} \left(\left[\partial_x^{s+1}E\right]^2 + \left[\partial_z^{s+1}E\right]^2 \right) \mathrm{d}x\mathrm{d}z \leq \int_{\mathbb{R}^2} \left(\frac{\left[\partial_x^{s+1}E\right]^2 + \left[\partial_z^{s+1}E\right]^2}{1+m} \right) \mathrm{d}x\mathrm{d}z < \infty,$$

and thus conclude that, for all $t \in [0, t_0]$,

$$\int_{\mathbb{R}^2} \left(\left[\partial_x^{s+1}E\right]^2 + \left[\partial_z^{s+1}E\right]^2 \right) \mathrm{d}x\mathrm{d}z < \infty. \tag{2.9}$$

It is also clear that the norm in (2.9) is a continuous function of t on $[0, t_0]$ so that the assertion (2.5) holds. To obtain (2.6), we use the bootstrapping argument for the second equation in the system (2.1) because the space $H^s(\mathbb{R}^2)$ is a Banach algebra for $s > 1$ (Proposition A). □

2.2. Continuation of local solutions of the quasilinear wave system

The following lemma tells us that a local solution of Lemma 1 can be continued as long as the solution, its time derivative, and its space gradient remain bounded in L^∞-norm. This result is similar to the blow-up criteria of solutions in other equations of fluid dynamics [2,20].

Lemma 2. *Local solution of the system* (2.1) *in Lemma* 1 *does not blow up as* $t \to t_0$ *if*

$$\sup_{t \in [0, t_0]} \left(\|E\|_{L^\infty} + \|\partial_t E\|_{L^\infty} + \|\nabla E\|_{L^\infty} \right) < \infty. \tag{2.10}$$

Proof. In order to verify the condition (2.10), we assume that $M_{1,2,3} < \infty$, where

$$M_1 := \sup_{t\in[0,t_0]} \|E\|_{L^\infty}, \quad M_2 := \sup_{t\in[0,t_0]} \|\nabla E\|_{L^\infty}, \quad M_3 := \sup_{t\in[0,t_0]} \|\partial_t E\|_{L^\infty},$$

and show that, for all $t \in [0, t_0]$,

$$\|E\|_{H^4}, \|\partial_t E\|_{H^3}, \left\|\partial_t^2 E\right\|_{H^2} < \infty.$$

To demonstrate this, we employ *a priori* energy bounds. For the sake of compactness, let us use short notation $E_x := \partial_x E$, $E_t := \partial_t E$ and so on for other derivatives of E and m.

Let us multiply the first equation of the system (2.1) by E_t and integrate by parts employing decay of $E_t E_x$ and $E_t E_z$ to zero as $|x|, |z| \to \infty$. The decay to zero is justified for the local solution of Lemma 1 with $s = 3$ by Sobolev's embeddings (Proposition B). Thus, we obtain

$$\frac{\mathrm{d}\mathcal{H}_1}{\mathrm{d}t} = \frac{1}{2}\int_{\mathbb{R}^2} E^2 E_t^2 \,\mathrm{d}x\mathrm{d}z \quad \Rightarrow \quad \frac{\mathrm{d}\mathcal{H}_1}{\mathrm{d}t} \le M_1^2 \mathcal{H}_1, \tag{2.11}$$

where we have used the second equation of the system (2.1) and introduced the first energy functional

$$\mathcal{H}_1 := \frac{1}{2}\int_{\mathbb{R}^2} \left((1+m) E_t^2 + E_x^2 + E_z^2\right) \mathrm{d}x\mathrm{d}z. \tag{2.12}$$

By Gronwall's inequality (Proposition D) and the fact that $m(x,z,t) \ge 0$ for all $(x,z) \in \mathbb{R}^2$, we obtain

$$\|E_x\|_{L^2}^2 + \|E_z\|_{L^2}^2 + \|E_t\|_{L^2}^2 \le 2\mathcal{H}_1 \le 2\mathcal{H}_1|_{t=0} e^{M_1^2 t} < \infty, \quad t \in [0, t_0].$$

By Lemma A for $p = 2$, we also control $\|E\|_{L^2}$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|E\|_{L^2} \le \|E_t\|_{L^2} \quad \Rightarrow \quad \|E\|_{L^2} \le t_0 \sup_{t\in[0,t_0]} \|E_t\|_{L^2} + (\|E\|_{L^2})|_{t=0} < \infty, \quad t \in [0, t_0].$$

We thus conclude that $E \in H^1(\mathbb{R}^2)$ and $E_t \in L^2(\mathbb{R}^2)$ for all $t \in [0, t_0]$.

Now, we perform the same procedure but differentiating the first equation of the system (2.1) with respect to x, multiplying it by E_{xt} and integrating over $(x,z) \in \mathbb{R}^2$. Repeating the same with z- and t-variables, we sum the results to obtain

$$\frac{\mathrm{d}\mathcal{H}_2}{\mathrm{d}t} = \frac{1}{2}\int_{\mathbb{R}^2} \left(E^2\left[E_{xt}^2 + E_{zt}^2 - E_{tt}^2\right] - E_{tt}\left[E_{xt} m_x + E_{zt} m_z\right]\right) \mathrm{d}x\mathrm{d}z, \tag{2.13}$$

where the second energy functional was introduced

$$\mathcal{H}_2 := \frac{1}{2}\int_{\mathbb{R}^2} \left((1+m) E_{tt}^2 + (2+m)\left[E_{xt}^2 + E_{zt}^2\right] + E_{xx}^2 + E_{zz}^2 + 2E_{xz}^2\right) \mathrm{d}x\mathrm{d}z, \tag{2.14}$$

and we have used the decay of $E_{xt}E_{xx}$, $E_{xt}E_{xz}$, $E_{zt}E_{zz}$, $E_{zt}E_{xz}$, $E_{tt}E_{xt}$ and $E_{tt}E_{zt}$ to zero as $|x|, |z| \to \infty$, which is justified for the local solution of Lemma 1 for $s = 3$. We have

$$\|E_{xx}\|_{L^2}^2 + \|E_{zz}\|_{L^2}^2 + 2\|E_{xz}\|_{L^2}^2 + 2\|E_{xt}\|_{L^2}^2 + 2\|E_{zt}\|_{L^2}^2 + \|E_{tt}\|_{L^2}^2 \le 2\mathcal{H}_2.$$

We shall now control $\mathcal{H}_2$ from Eq. (2.13). The terms in (2.13) with $E^2E_{xt}^2$, $E^2E_{zt}^2$ and $E^2E_{tt}^2$ are controlled by a multiple of $M_1^2\mathcal{H}_2$. Additionally, we need to bound $\|m\|_{L^\infty}$ and $\|\nabla m\|_{L^\infty}$. By Corollary A for $p = \infty$, we have

$$\|m\|_{L^\infty} \le t_0 \sup_{t\in[0,t_0]} \|m_t\|_{L^\infty} \le t_0 M_1^2 \tag{2.15}$$

and

$$\|\nabla m\|_{L^\infty} \le t_0 \sup_{t\in[0,t_0]} \|\nabla m_t\|_{L^\infty} \le 2t_0 M_1 M_2, \quad t \in [0, t_0], \tag{2.16}$$

where we have used the initial condition $m|_{t=0} = 0$ and the second equation of the system (2.1).

By the triangle and Cauchy–Schwarz inequalities, we have

$$\frac{\mathrm{d}\mathcal{H}_2}{\mathrm{d}t} \leq M_1\left(M_1 + 2t_0 M_2\right)\mathcal{H}_2, \quad \Rightarrow \quad \mathcal{H}_2 \leq \mathcal{H}_2|_{t=0} e^{M_1(M_1+2t_0M_2)t} < \infty, \quad t \in [0, t_0].$$

Thus, we deduce that $E \in H^2\left(\mathbb{R}^2\right)$, $E_t \in H^1\left(\mathbb{R}^2\right)$, and $E_{tt} \in L^2\left(\mathbb{R}^2\right)$ for all $t \in [0, t_0]$.

We continue in the same manner as before, acting on the first equation of the system (2.1) by the operator $E_{xxt}\partial_x^2 + E_{zzt}\partial_z^2 + E_{ttt}\partial_t^2$ and integrating the result in (x, z) over $\mathbb{R}^2$ by parts to reduce the expression to first-order derivatives of m only. At the end, we obtain a functional that is not positive definite. Its boundedness does not yield a bound on the norms of derivatives of E it includes. To remedy the situation, we add $\int_{\mathbb{R}^2} \left(m_x^2 + m_z^2\right) E_{tt}^2 \,\mathrm{d}x\mathrm{d}z$ to the energy functional thus obtained and compute the balance equation:

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}_3}{\mathrm{d}t} = \frac{1}{2}\int_{\mathbb{R}^2}\Big(& E^2\left[E_{xxt}^2 + E_{zzt}^2 - 3E_{ttt}^2\right] - 2m_x\left[E_{xxt}E_{xtt} + E_{xxx}E_{ttt}\right] \\ & -2m_z\left[E_{zzt}E_{ztt} + E_{zzz}E_{ttt}\right] - 4E_{tt}E\left[E_{ttt}E_t + E_{xxx}E_x + E_{zzz}E_z\right] \\ & +8EE_{tt}^2\left[E_x m_x + E_z m_z\right] + 4E_{tt}E_{ttt}\left[m_x^2 + m_z^2\right]\Big)\,\mathrm{d}x\mathrm{d}z, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned}\mathcal{H}_3 := \frac{1}{2}\int_{\mathbb{R}^2}\Big(& (1+m)\left[E_{ttt}^2 + E_{xxt}^2 + E_{zzt}^2\right] + E_{xtt}^2 + E_{ztt}^2 + \frac{1}{2}\left(E_{xxx}^2 + E_{zzz}^2\right) \\ & +E_{xxz}^2 + E_{xzz}^2 + \frac{1}{2}\left[E_{xxx} - 2m_x E_{tt}\right]^2 + \frac{1}{2}\left[E_{zzz} - 2m_z E_{tt}\right]^2\Big)\,\mathrm{d}x\mathrm{d}z, \end{aligned} \tag{2.18}$$

so that

$$\begin{aligned} & \|E_{xxx}\|_{L^2}^2 + \|E_{zzz}\|_{L^2}^2 + 2\|E_{xxz}\|_{L^2}^2 + 2\|E_{xzz}\|_{L^2}^2 + 2\|E_{xxt}\|_{L^2}^2 + 2\|E_{zzt}\|_{L^2}^2 \\ & \quad +2\|E_{ttt}\|_{L^2}^2 + 2\|E_{xtt}\|_{L^2}^2 + 2\|E_{ztt}\|_{L^2}^2 \leq 4\mathcal{H}_3. \end{aligned}$$

In deriving Eq. (2.17), we have used the decay of $E_{xxx}E_{xxt}$, $E_{xxz}E_{xxt}$, $E_{zzz}E_{zzt}$, $E_{xzz}E_{zzt}$, $E_{ttt}E_{xtt}$, $E_{ttt}E_{ztt}$, $m_xE_{xxt}E_{tt}$, and $m_zE_{zzt}E_{tt}$ to zero as $|x| \to \infty$, $|z| \to \infty$. This decay can be obtained by working with approximating sequences as follows.

Let us consider an approximation of the initial conditions E_0, E_1 by the sequences of functions $\left\{E_0^{(n)}\right\}_{n=1}^{\infty} \in H^5\left(\mathbb{R}^2\right)$, $\left\{E_1^{(n)}\right\}_{n=1}^{\infty} \in H^4\left(\mathbb{R}^2\right)$, respectively. Then, by Lemma 1 with $s = 4$, the corresponding sequence of local solutions will be

$$E^{(n)} \in C\left([0, t_0], H^5\left(\mathbb{R}^2\right)\right) \cap C^1\left([0, t_0], H^4\left(\mathbb{R}^2\right)\right) \cap C^2\left([0, t_0], H^3\left(\mathbb{R}^2\right)\right).$$

The decay assumptions are valid for the approximate solution $E^{(n)}$ by Sobolev's embeddings.

Because the space $H^5\left(\mathbb{R}^2\right)$ is dense in $H^4\left(\mathbb{R}^2\right)$ and so is $H^4\left(\mathbb{R}^2\right)$ in $H^3\left(\mathbb{R}^2\right)$, we have

$$\left\|E_0^{(n)} - E_0\right\|_{H^4} \to 0, \quad \left\|E_1^{(n)} - E_1\right\|_{H^3} \to 0 \quad \text{as } n \to \infty,$$

and hence, by the continuous dependence of the solution on the initial data in Lemma 1, we have

$$\left\|E^{(n)} - E\right\|_{H^4} \to 0, \quad \left\|E_t^{(n)} - E_t\right\|_{H^3} \to 0 \quad \text{as } n \to \infty,$$

that holds for all $t \in [0, t_0]$. This approximation argument furnishes the required decay of solutions at infinity in the justification of the energy balance (2.17).

Using (2.16), we estimate the m-dependent terms in (2.17) as follows:

$$\int_{\mathbb{R}^2} m_x \left[E_{xxt}E_{xtt} + E_{xxx}E_{ttt}\right] \mathrm{d}x\mathrm{d}z \leq \|\nabla m\|_{L^\infty} \left(\|E_{xxt}\|_{L^2} \|E_{xtt}\|_{L^2} + \|E_{xxx}\|_{L^2} \|E_{ttt}\|_{L^2}\right) \leq 8t_0 M_1 M_2 \mathcal{H}_3,$$

$$\int_{\mathbb{R}^2} E_{tt}E_{ttt}m_x^2 \,\mathrm{d}x\mathrm{d}z \leq \|\nabla m\|_{L^\infty}^2 \|E_{tt}\|_{L^2} \|E_{ttt}\|_{L^2} \leq 8t_0^2 M_1^2 M_2^2 \mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2},$$

and

$$\int_{\mathbb{R}^2} EE_x E_{tt}^2 m_x \,\mathrm{d}x\mathrm{d}z \leq M_1 M_2 \|\nabla m\|_{L^\infty} \|E_{tt}\|_{L^2}^2 \leq 4t_0 M_1^2 M_2^2 \mathcal{H}_2.$$

Similar estimates are obtained for the z-derivatives terms.

The estimates of the m-independent terms in (2.17) are straightforward as follows:

$$\int_{\mathbb{R}^2} E^2 \left[E_{xxt}^2 + E_{zzt}^2 - 3E_{ttt}^2\right] \mathrm{d}x\mathrm{d}z \leq M_1^2 \mathcal{H}_3$$

and

$$\int_{\mathbb{R}^2} EE_{tt} \left[E_{xxx}E_x + E_{zzz}E_z + E_{ttt}E_t\right] \mathrm{d}x\mathrm{d}z \leq M_1 \left(2M_2 + M_3\right) \mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2}.$$

Combining all results together, we write

$$\frac{\mathrm{d}\mathcal{H}_3}{\mathrm{d}t} \leq M_1 \left(M_1 + 16t_0 M_2\right)\mathcal{H}_3 + 2M_1 \left(2M_2 + M_3 + 16t_0^2 M_1 M_2^2\right) \mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2} + 32t_0 M_1^2 M_2^2 \mathcal{H}_2.$$

Using now the inequality $\mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2} \leq \frac{1}{2}\left(\mathcal{H}_2 + \mathcal{H}_3\right)$, we obtain

$$\frac{\mathrm{d}\mathcal{H}_3}{\mathrm{d}t} \leq F\mathcal{H}_3 + G\mathcal{H}_2,$$

where

$$F := M_1 \left(M_1 + 18t_0 M_2 + t_0 M_3 + 16t_0^3 M_1 M_2^2\right),$$
$$G := M_1 \left(2M_2 + M_3 + 16t_0^2 M_1 M_2^2\right) + 32t_0 M_1^2 M_2^2.$$

Then, by Gronwall's inequality (A.9), for $t \in [0, t_0]$, we obtain

$$\mathcal{H}_3 \leq \left(\mathcal{H}_3|_{t=0} + Gt_0 \sup_{t\in[0,t_0]} \mathcal{H}_2\right) e^{tF} < \infty, \quad t \in [0, t_0].$$

Thus, we deduce that $E \in H^3\left(\mathbb{R}^2\right)$, $E_t \in H^2\left(\mathbb{R}^2\right)$, and $E_{tt} \in H^1\left(\mathbb{R}^2\right)$ for all $t \in [0, t_0]$.

We proceed to obtain the final energy estimates. We act on the first equation of the system (2.1) by the operator

$$E_{xxxt}\partial_x^3 + E_{zzzt}\partial_z^3 + E_{xttt}\partial_x\partial_t^2 + E_{zttt}\partial_z\partial_t^2 + E_{tttt}\partial_t^3$$

and integrate the result in (x, z) over $\mathbb{R}^2$. Following the same steps as in the previous energy level computations, we can introduce the positive definite energy functional

$$\begin{aligned}\mathcal{H}_4 := \frac{1}{2}\int_{\mathbb{R}^2} \Big[&(1+m)\left(E_{xxxt}^2 + E_{zzzt}^2 + E_{xttt}^2 + E_{zttt}^2 + E_{tttt}^2\right) + \frac{1}{2}\left(E_{xxxx}^2 + E_{zzzz}^2\right) \\ &+2E_{xxzz}^2 + 2E_{xztt}^2 + E_{xxtt}^2 + E_{zztt}^2 + E_{xttt}^2 + E_{zttt}^2 + \frac{1}{2}\left(E_{xxxx} - 2m_{xx}E_{tt}\right)^2 \\ &+ \frac{1}{2}\left(E_{zzzz} - 2m_{zz}E_{tt}\right)^2\Big] \mathrm{d}x\mathrm{d}z \end{aligned} \quad (2.19)$$

to obtain

$$\frac{\mathrm{d}\mathcal{H}_4}{\mathrm{d}t} = \frac{1}{2}\int\limits_{\mathbb{R}^2} \left[E^2\left(E_{xxxt}^2 + E_{zzzt}^2 - 3E_{xttt}^2 - 3E_{zttt}^2 - 4E_{tttt}^2\right) - 4E_{tt}\left(E_x^2 E_{xxxx} + E_z^2 E_{zzzz} + E_t^2 E_{tttt}\right)\right.$$
$$- 4EE_{tt}\left(E_{xx}E_{xxxx} + E_{zz}E_{zzzz} + E_{tt}E_{tttt}\right) - 4m_{xx}\left(E_{xtt}E_{xxxt} + E_{ttt}E_{xxxx}\right)$$
$$- 4m_{zz}\left(E_{ztt}E_{zzzt} + E_{ttt}E_{zzzz}\right) - 6m_x E_{xxtt}E_{xxxt} - 6m_z E_{zztt}E_{zzzt} - 12EE_tE_{ttt}E_{tttt}$$
$$\left. + 4E_{ttt}E_{tt}\left(m_{xx}^2 + m_{zz}^2\right) + 8E_{tt}^2\left(m_{xx}\left(E_x^2 + EE_{xx}\right) + m_{zz}\left(E_z^2 + EE_{zz}\right)\right)\right] \mathrm{d}x\mathrm{d}z. \qquad (2.20)$$

This computation is valid under assumption on decay to zero of $E_{xxxt}E_{xxxx}$, $E_{xxxt}E_{xxxz}$, $E_{zzzt}E_{zzzz}$, $E_{zzzt}E_{xzzz}$, $E_{tttt}E_{xttt}$, $E_{tttt}E_{zttt}$, $E_{xttt}E_{xxtt}$, $E_{xttt}E_{xztt}$, $E_{zttt}E_{zztt}$, $E_{zttt}E_{xztt}$, $m_{xx}E_{xxtt}E_{tt}$, and $m_{zz}E_{zztt}E_{tt}$ as $|x| \to \infty$, $|z| \to \infty$. This required decay can be justified by the approximation argument for a sequence of local solutions of Lemma 1 with $s = 5$ as done in the previous computations of the balance equation (2.17). We have the control:

$$\|E_{xxxx}\|_{L^2}^2 + \|E_{zzzz}\|_{L^2}^2 + 4\|E_{xxzz}\|_{L^2}^2 + 2\|E_{xxxt}\|_{L^2}^2 + 2\|E_{zzzt}\|_{L^2}^2$$
$$+2\|E_{xxtt}\|_{L^2}^2 + 2\|E_{zztt}\|_{L^2}^2 + 4\|E_{xztt}\|_{L^2}^2 \leq 4\mathcal{H}_4. \qquad (2.21)$$

Hence, by Sobolev's embeddings, we have

$$\|E_{xx}\|_{L^\infty} \leq C_0\left(\|\Delta E_{xx}\|_{L^2} + \|E_{xx}\|_{L^2}\right) \leq \sqrt{2}C_0\left(\mathcal{H}_4^{1/2} + \mathcal{H}_2^{1/2}\right),$$

for some $C_0 > 0$. Using this estimate and Corollary A, we obtain from the second equation of the system (2.1) for $t \in [0, t_0]$,

$$\|m_{xx}\|_{L^\infty} \leq t_0 \sup_{[0,t]}\|m_{txx}\|_{L^\infty} \leq 2t_0M_2^2 + 2\sqrt{2}C_0t_0M_1\left(\sup_{[0,t]}\mathcal{H}_4^{1/2} + \sup_{t\in[0,t_0]}\mathcal{H}_2^{1/2}\right).$$

Similar estimates hold for the z-derivatives terms. Lengthy calculations result in the inequality

$$\frac{\mathrm{d}\mathcal{H}_4}{\mathrm{d}t} \leq I\mathcal{H}_4 + J\sup_{t\in[0,t_0]}\mathcal{H}_4 + L, \qquad (2.22)$$

where I, J, L are some coefficients that depend on t_0, M_1, M_2, M_3, $\sup_{t\in[0,t_0]}\mathcal{H}_2$ and $\sup_{t\in[0,t_0]}\mathcal{H}_3$. Inequality (2.22) can be integrated as follows:

$$\sup_{t\in[0,t_0]}\mathcal{H}_4 \leq \mathcal{H}_4|_{t=0} + t_0L + (I+J)\,t_0\sup_{t\in[0,t_0]}\mathcal{H}_4.$$

By the integral form of Gronwall's inequality, we hence estimate

$$\sup_{t\in[0,t_0]}\mathcal{H}_4 \leq \left(\mathcal{H}_4|_{t=0} + t_0L\right)e^{t(I+J)} < \infty.$$

Now, since

$$\|E_{xxxz}\|_{L^2}^2, \|E_{xzzz}\|_{L^2}^2 \leq \frac{1}{2}\|E_{xxxx}\|_{L^2}^2 + \frac{1}{2}\|E_{xxzz}\|_{L^2}^2$$

and

$$\|E_{xxzt}\|_{L^2}^2, \|E_{xzzt}\|_{L^2}^2 \leq \frac{1}{2}\|E_{xxxt}\|_{L^2}^2 + \frac{1}{2}\|E_{zzzt}\|_{L^2}^2,$$

which is a result of straightforward estimates on the Fourier transform side, we conclude that $E \in H^4\left(\mathbb{R}^2\right)$, $E_t \in H^3\left(\mathbb{R}^2\right)$ and $E_{tt} \in H^2\left(\mathbb{R}^2\right)$ for all $t \in [0, t_0]$. □

Remark 1. To eliminate finite-time blow-up of the component m in H^4-norm, we can use the estimate of E in Lemma 2 and the Banach algebra property in Proposition A applied to the second equation of the system (2.1).

2.3. Local well-posedness of the NLS system

To study local well-posedness of the NLS system (1.9)–(1.10), we shall work with the rescaled equations

$$\begin{cases} \partial_X^2 A + i\left(\partial_T A + \partial_Z A\right) + MA = 0, \\ \partial_T M = \left|A\right|^2, \end{cases} \qquad (X,Z) \in \mathbb{R}^2, \quad T \in \mathbb{R}_+, \tag{2.23}$$

subject to the initial data $A|_{T=0} = A_0 \in H^s\left(\mathbb{R}^2\right)$, $M|_{T=0} = 0$, for some integer $s \geq 2$. The following lemma gives local well-posedness result.

Lemma 3. *For any integer $s \geq 2$ and any $\delta > 2\left\|A_0\right\|_{H^s(\mathbb{R}^2)}$, there exist a positive constant T_0 and a unique solution $A \in C\left(\left[0,T_0\right], H^s\left(\mathbb{R}^2\right)\right) \cap C^1\left(\left[0,T_0\right], H^{s-2}\left(\mathbb{R}^2\right)\right)$ to the NLS system (2.23) such that $A|_{T=0} = A_0$ and $\sup_{T\in[0,T_0]} \left\|A\right\|_{H^s(\mathbb{R}^2)} \leq \delta$.*

Proof. Let us take Fourier transform in both spatial variables and denote

$$\widehat{MA}(\xi,\eta,T) := \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} M(X,Z,T)A(X,Z,T)e^{i(\xi X + \eta Z)}\,\mathrm{d}x\mathrm{d}z.$$

The first equation of the NLS system (2.23) then becomes

$$\partial_T \hat{A} = i\left(-\xi^2 + \eta\right)\hat{A} + i\widehat{MA},$$

which leads to the integral equation

$$\hat{A}\left(\xi,\eta,T\right) = \hat{A}_0\left(\xi,\eta\right)e^{i\left(-\xi^2+\eta\right)T} + i\int\limits_0^T e^{i\left(-\xi^2+\eta\right)(T-\tau)}\widehat{MA}\left(\xi,\eta,\tau\right)\,\mathrm{d}\tau. \tag{2.24}$$

Introduce the Schrödinger kernel

$$S_T\left(X\right) := \frac{1}{\sqrt{4\pi T}}e^{-\frac{i\pi}{4}}e^{\frac{iX^2}{4T}},$$

such that

$$\mathcal{F}^{-1}\left[e^{i\left(-\xi^2+\eta\right)T}\right] = S_T\left(X\right)\delta\left(T - Z\right).$$

Using the inverse Fourier transform of (2.24), we obtain the integral equation

$$A\left(X,Z,T\right) = S_T\left(X\right)\star A_0\left(X,Z-T\right) + i\int\limits_0^T S_{T-\tau}\left(X\right)\star\left[M\left(X,Z-T+\tau,\tau\right)A\left(X,Z-T+\tau,\tau\right)\right]\mathrm{d}\tau,$$

where $\star$ stands for convolution in X-variable.

Making use of the second equation in the NLS system (2.23) and $M|_{T=0} = 0$, we can rewrite the integral equation as an operator equation

$$A\left(X,Z,T\right) = K\left[A\left(X,Z,T\right)\right], \tag{2.25}$$

where

$$K\left[A\right] := S_T\left(X\right)\star A_0\left(X,Z-T\right) + i\int\limits_0^T S_{T-\tau}\left(X\right)\star\left[A\left(X,Z-T+\tau,\tau\right)\int\limits_0^\tau\left|A\left(X,Z-T+\tau,\tilde{\tau}\right)\right|^2 d\tilde{\tau}\right]\mathrm{d}\tau.$$

Existence and uniqueness of solutions to Eq. (2.25) are obtained by applying the Banach fixed-point theorem (Proposition E). We need to show that conditions of the fixed-point theorem are fulfilled in a closed ball of radius δ in the space $C\left([0,T_0],H^s\left(\mathbb{R}^2\right)\right)$ for some $\delta > 0$ and $T_0 > 0$ as well as for any $s \geq 2$:

$$\bar{\mathcal{B}}_\delta := \left\{ f \in C\left([0,T_0],H^s\left(\mathbb{R}^2\right)\right) : \sup_{T\in[0,T_0]} \|f\|_{H^s(\mathbb{R}^2)} \leq \delta \right\}. \tag{2.26}$$

In other words, we need to show that $\bar{\mathcal{B}}_\delta$ is an invariant subspace of the operator K, that is, for any $A \in \bar{\mathcal{B}}_\delta \subset C\left([0,T_0],H^s\left(\mathbb{R}^2\right)\right)$, we have

$$\sup_{T\in[0,T_0]} \|K[A]\|_{H^s(\mathbb{R}^2)} \leq \delta, \tag{2.27}$$

for suitable choice of $\delta > 0$ and $T_0 > 0$. We also need to show that K is a contractive operator in the sense that there is $q \in (0,1)$ such that for any $A^{(1)}$, $A^{(2)} \in \bar{\mathcal{B}}_\delta$

$$\sup_{T\in[0,T_0]} \left\| K\left[A^{(1)}\right] - K\left[A^{(2)}\right] \right\|_{H^s(\mathbb{R}^2)} \leq q \sup_{T\in[0,T_0]} \left\| A^{(1)} - A^{(2)} \right\|_{H^s(\mathbb{R}^2)}. \tag{2.28}$$

To choose $\delta > 0$ and $T_0 > 0$ such that both conditions (2.27)–(2.28) are satisfied, we proceed with analysis on the Fourier transform side using (2.24) rather than (2.25).

We start by showing (2.27). Let $A \in \bar{\mathcal{B}}_\delta$, that is, $\sup_{T\in[0,T_0]} \|A\|_{H^s(\mathbb{R}^2)} \leq \delta$. Then, applying Plancherel's theorem and Minkowski's integral inequality to (2.24), we obtain

$$\begin{aligned}
\sup_{T\in[0,T_0]} \|K[A]\|_{H^s(\mathbb{R}^2)} &= \sup_{T\in[0,T_0]} \left\| \left(1+\xi^2+\eta^2\right)^{s/2} \hat{K}[A] \right\|_{L^2(\mathbb{R}^2)} \\
&\leq \left\| \left(1+\xi^2+\eta^2\right)^{s/2} \hat{A}_0 \right\|_{L^2(\mathbb{R}^2)} + \sup_{T\in[0,T_0]} \int_0^T \left\| \left(1+\xi^2+\eta^2\right)^{s/2} \widehat{MA} \right\|_{L^2(\mathbb{R}^2)} \mathrm{d}\tau \\
&= \|A_0\|_{H^s(\mathbb{R}^2)} + \sup_{T\in[0,T_0]} \int_0^T \|MA\|_{H^s(\mathbb{R}^2)} \mathrm{d}\tau.
\end{aligned}$$

By Proposition A and Corollary B, we arrive at the bounds

$$\begin{aligned}
\sup_{T\in[0,T_0]} \|K[A]\|_{H^s(\mathbb{R}^2)} &\leq \|A_0\|_{H^s(\mathbb{R}^2)} + C_s^2 T_0^2 \sup_{T\in[0,T_0]} \|A\|^3_{H^s(\mathbb{R}^2)} \\
&\leq \|A_0\|_{H^s(\mathbb{R}^2)} + C_s^2 T_0^2 \delta^3,
\end{aligned}$$

for some constant $C_s > 0$. If $\delta \geq 2\|A_0\|_{H^s(\mathbb{R}^2)}$ and $T_0 \leq \frac{1}{\sqrt{2}C_s\delta}$, then both terms become less or equal to $\delta/2$ which furnishes (2.27).

Now, we proceed with showing (2.28). We write

$$M\left(A^{(1)}\right) = M\left(A^{(2)}\right) - \left(M\left(A^{(2)}\right) - M\left(A^{(1)}\right)\right)$$

and using the triangle inequality, the Banach algebra property of $H^s\left(\mathbb{R}^2\right)$, and the same arguments as above, we obtain

$$\begin{aligned}
&\sup_{T\in[0,T_0]}\left\|K\left[A^{(1)}\right]-K\left[A^{(2)}\right]\right\|_{H^s(\mathbb{R}^2)}\\
&\quad\le \sup_{T\in[0,T_0]}\int_0^T\left\|M\left(A^{(1)}\right)A^{(1)}-M\left(A^{(2)}\right)A^{(2)}\right\|_{H^s}\mathrm{d}\tau\\
&\quad\le C_s^2T_0^2\sup_{T\in[0,T_0]}\left[\left\|A^{(2)}\right\|_{H^s}^2\left\|A^{(1)}-A^{(2)}\right\|_{H^s}+\left\|A^{(1)}\right\|_{H^s}\left(\left\|A^{(1)}\right\|_{H^s}+\left\|A^{(2)}\right\|_{H^s}\right)\left\|A^{(1)}-A^{(2)}\right\|_{H^s}\right]\\
&\quad\le 3C_s^2T_0^2\delta^2\sup_{T\in[0,T_0]}\left\|A^{(1)}-A^{(2)}\right\|_{H^s(\mathbb{R}^2)}.
\end{aligned}$$

From here, contraction of operator K is achieved if $T_0<\frac{1}{\sqrt{3}C_s\delta}$. Combining this with the previous condition, we conclude that the choice

$$\delta>2\left\|A_0\right\|_{H^s(\mathbb{R}^2)},\quad T_0\le\frac{1}{\sqrt{3}C_s\delta}$$

leads to the existence of unique solution A of Eq. (2.25) in the ball (2.26).

Then, expressing $\partial_T A$ from the first equation of the system (2.23), the bootstrapping argument gives $A\in C^1\left(\left[0,T_0\right],H^{s-2}\left(\mathbb{R}^2\right)\right)$. □

Remark 2. Tracing the proof, it is straightforward to see that the same result holds in the presence of inhomogeneous terms in the system (2.23) providing these terms belong to the space $C\left(\left[0,T_0\right],H^s\left(\mathbb{R}^2\right)\right)$.

3. Rigorous justification analysis

We shall here prove Theorem 1 by using near-identity transformations and *a priori* energy estimates.

3.1. Near-identity transformations

Smallness of error terms $U(x,z,t)$ and $N(x,z,t)$ in the decompositions (1.12)–(1.13) hinges on smallness of the right-hand side terms in the system (1.14)–(1.15). The right-hand side terms can be made smaller by performing near-identity transformations.

Let us start with the source term $\epsilon^6\left(R_6^{(U)}\right)_{\omega_0}$ in Eq. (1.14) and introduce

$$U_1(x,z,t):=U(x,z,t)-\epsilon^4\left(F(X,Z,T)\right)_{\omega_0},\tag{3.1}$$

where $F(X,Z,T)$ will be chosen later. Eliminating $U(x,z,t)$ from (1.14), we obtain

$$\partial_x^2U_1+\partial_z^2U_1-\left(1+\epsilon^2M+N\right)\partial_t^2U_1=-\epsilon^2\left(\tilde{R}_2^{(U)}\right)_{\omega_0}N-\epsilon^6\left(\tilde{R}_6^{(U)}\right)_{\omega_0}-\epsilon^8\left(\tilde{R}_8^{(U)}\right)_{\omega_0},\tag{3.2}$$

where

$$\begin{aligned}
\tilde{R}_2^{(U)}&=R_2^{(U)}+\epsilon^2\left(\omega_0^2F+2i\omega_0\epsilon^2\partial_TF-\epsilon^4\partial_T^2F\right),\\
\tilde{R}_6^{(U)}&=\partial_X^2F+2i\omega_0\left(\partial_ZF+\partial_TF\right)+\omega_0^2MF-\left(\partial_T^2A-\partial_Z^2A-2i\omega_0M\partial_TA\right),\\
\tilde{R}_8^{(U)}&=\partial_Z^2F-\partial_T^2F-M\partial_T^2A+2i\omega_0M\partial_TF-\epsilon^2M\partial_T^2F.
\end{aligned}$$

The $\mathcal{O}\left(\epsilon^6\right)$ source term is eliminated (that is, $\tilde{R}_6^{(U)} = 0$) providing that $F(X,Z,T)$ solves the linear inhomogeneous Schrödinger equation

$$\partial_X^2 F + 2i\omega_0\left(\partial_Z F + \partial_T F\right) + \omega_0^2 M F = \partial_T^2 A - \partial_Z^2 A - 2i\omega_0 M \partial_T A. \tag{3.3}$$

Hence, Eq. (3.2) for $U_1(x,z,t)$ has a $\mathcal{O}\left(\epsilon^8\right)$ source term. Generally, such transformation can be repeated k times to have a source term of order $\mathcal{O}\left(\epsilon^{6+2k}\right)$, but one application of the transformation (3.1) ($k = 1$) will be sufficient for us to close the estimates.

Now, we proceed with Eq. (1.15) treating the first two terms in the right-hand side separately. To remove the $\mathcal{O}\left(\epsilon^4\right)$ source term, we introduce

$$N_1 := N - \epsilon^4\left(\frac{A^2}{2i\omega_0}\right)_{2\omega_0} \tag{3.4}$$

and obtain the equation for $N_1(x,z,t)$ with the $\mathcal{O}\left(\epsilon^6\right)$ source term:

$$\partial_t N_1 = -\epsilon^6\left(\frac{A\partial_T A}{i\omega_0}\right)_{2\omega_0} + 2\epsilon^2 (A)_{\omega_0} U + U^2. \tag{3.5}$$

In a similar fashion, this transformation can be repeated n times to get the $\mathcal{O}\left(\epsilon^{4+2n}\right)$ source term. We will need two transformations of the type (3.4) ($n = 2$) to move the residual term to the $\mathcal{O}(\epsilon^8)$ order, which will be sufficient for us to close the estimates.

To improve the second term in the right-hand side of (1.15), we perform another type of the near-identity transformation

$$N_2 := N - 2\epsilon^2\left(\frac{A}{i\omega_0}\right)_{\omega_0} U, \tag{3.6}$$

in which case we have

$$\partial_t N_2 = \epsilon^4\left(A^2\right)_{2\omega_0} - 2\epsilon^2\left(\frac{A}{i\omega_0}\right)_{\omega_0}\partial_t U - 2\epsilon^4\left(\frac{\partial_T A}{i\omega_0}\right)_{\omega_0} U + U^2. \tag{3.7}$$

This transformation moves the linear term in U to the $\mathcal{O}\left(\epsilon^4\right)$ order, whereas the $\mathcal{O}\left(\epsilon^2\right)$ term depends now on $\partial_t U$, which norm is expected to be smaller. Note that, an iteration of this latter transformation is not effective because we do not anticipate the norm of $\partial_t^2 U$ to be smaller compared to the norm of $\partial_t U$. The two near-identity transformations (3.4) and (3.6) can be combined in a straightforward way, however, putting together near-identity transformations (3.1), (3.4), and (3.6) should be done more carefully due to intertwining structure of the equations.

Including the third-harmonic term in the relevant transformations because of the nonlinear terms produced in the system (1.14)–(1.15), we write the resulting near-identity transformations in the form

$$V := U - \epsilon^4 (B)_{\omega_0} - \epsilon^4 (D)_{3\omega_0}, \tag{3.8}$$

$$\begin{aligned} P := {} & N - \epsilon^4 N_0 + 2\epsilon^2\left(\frac{A}{i\omega_0}\right)_{\omega_0} V + \epsilon^4\left(\frac{A^2}{2i\omega_0}\right)_{2\omega_0} \\ & + \epsilon^6\left[\frac{2\left(A\bar{B} - \bar{A}B\right)}{i\omega_0} - \left(\frac{A\partial_T A}{2\omega_0^2} - \frac{AB + \bar{A}D}{i\omega_0}\right)_{2\omega_0} + \left(\frac{AD}{2i\omega_0}\right)_{4\omega_0}\right]. \end{aligned} \tag{3.9}$$

Here, bar denotes complex conjugation, and $B(X,Z,T)$, $D(X,Z,T)$, $N_0(X,Z,T)$ solve the following linear inhomogeneous equations

$$\partial_X^2 B + 2i\omega_0\left(\partial_Z B + \partial_T B\right) + \omega_0^2 M B = \partial_T^2 A - \partial_Z^2 A - 2i\omega_0 M \partial_T A - \frac{i\omega_0}{2} A^2 \bar{A}, \tag{3.10}$$

$$\partial_X^2 D + 2i\omega_0\left(\partial_Z D + \partial_T D\right) + \omega_0^2 M D = -\frac{i\omega_0}{2} A^3, \tag{3.11}$$

and

$$\partial_T N_0 = 2\left(A\bar{B} + \bar{A}B\right). \tag{3.12}$$

As a result of the transformations (3.8)–(3.9) and the relations (3.10)–(3.12), the system (1.14)–(1.15) transforms to the system

$$\partial_x^2 V + \partial_z^2 V - \left(1 + \epsilon^2 M + N\right)\partial_t^2 V = -\epsilon^2\omega_0^2 (A)_{\omega_0} P + 2i\omega_0\epsilon^4 |A|^2 V - \epsilon^8 R_8^{(V)} \tag{3.13}$$

and

$$\partial_t P = \epsilon^8 R_8^{(P)} + 2\epsilon^4 R_4^{(P)} V + 2\epsilon^2 \left(\frac{A}{i\omega_0}\right)_{\omega_0} \partial_t V + V^2, \tag{3.14}$$

where

$$R_4^{(P)} = \left(\frac{\partial_T A}{i\omega_0} + B\right)_{\omega_0} + (D)_{3\omega_0},$$

$$R_8^{(P)} = -\frac{\partial_T\left(\bar{A}B - A\bar{B}\right)}{2i\omega_0} + \left(B\bar{D} - \frac{\partial_T\left(A\partial_T A\right)}{2\omega_0^2} + \frac{\partial_T\left(AB + \bar{A}D\right)}{i\omega_0}\right)_{2\omega_0} + \left(BC + \frac{\partial_T (AD)}{2i\omega_0}\right)_{4\omega_0},$$

$$R_8^{(V)} = \left(\frac{i\omega_0}{2}\left[2|A|^2 B + A^2\bar{B} + 7\bar{A}D + 4M\partial_T B\right] + \frac{|A|^2\partial_T A}{2} + A^2\partial_T\bar{A} - M\partial_T^2 A + \partial_Z^2 B - \partial_T^2 B\right)_{\omega_0}$$
$$- \left(\frac{i\omega_0}{2}\left[|A|^2 D + A^2 B - 12M\partial_T D\right] + \frac{A^2\partial_T A}{2} - \partial_Z^2 D + \partial_T^2 D\right)_{3\omega_0} + 3\left(i\omega_0 A^2 D\right)_{5\omega_0}.$$

This system of residual equations is a starting point in our justification analysis.

3.2. A priori energy estimates

We now proceed with the estimates of the error terms $U(x,z,t)$, $N(x,z,t)$ in the decompositions (1.12)–(1.13) given sufficiently smooth initial data. The amplitudes A and M change on the temporal scale of $T = \epsilon^2 t$ on $[0, T_0]$. Therefore, the validity of approximation needs to be justified for all $t \in \left[0, T_0/\epsilon^2\right]$. We would like to prove that there are α_0, $\beta_0 > 0$ such that

$$\sup_{t\in[0,T_0/\epsilon^2]} \|U(\cdot,\cdot,t)\|_{L^2} = \mathcal{O}(\epsilon^{\beta_0}), \qquad \sup_{t\in[0,T_0/\epsilon^2]} \|N(\cdot,\cdot,t)\|_{L^2} = \mathcal{O}(\epsilon^{\alpha_0}). \tag{3.15}$$

Because of the $\mathcal{O}(\epsilon^{1/2})$ order of the leading-order terms in (1.20) and (1.21), the error terms in the decompositions (1.12)–(1.13) are smaller than the leading-order terms in L^2-norm if α_0, $\beta_0 > \frac{1}{2}$. We intend to prove that the estimates can be closed with $\alpha_0 = \beta_0 = \frac{5}{2}$.

We shall use index notation for partial derivatives such as $E_x := \partial_x E$, $E_t := \partial_t E$, and so on. We shall also employ subscript notations such as $\|\cdot\|_{L^2_{X,Z}}$, $\nabla_{X,Z}$, and $\Delta_{X,Z}$ when necessary to emphasize that the norms or derivatives are computed with respect to slow variables X, Z. A generic positive constant is denoted by C.

Using the near-identity transformations (3.8) and (3.9), under assumptions B, D, $N_0 \in L^2\left(\mathbb{R}^2\right)$, $A \in L^\infty\left(\mathbb{R}^2\right) \cap L^2\left(\mathbb{R}^2\right)$, $R_6^{(M)} \in L^2\left(\mathbb{R}^2\right)$, we can see that

$$\sup_{t\in[0,T_0/\epsilon^2]} \|U(\cdot,\cdot,t)\|_{L^2} \le \sup_{t\in[0,T_0/\epsilon^2]} \|V(\cdot,\cdot,t)\|_{L^2} + C\epsilon^{5/2} \tag{3.16}$$

and

$$\sup_{t\in[0,T_0/\epsilon^2]} \|N(\cdot,\cdot,t)\|_{L^2} \le \sup_{t\in[0,T_0/\epsilon^2]} \|P(\cdot,\cdot,t)\|_{L^2} + C\epsilon^2 \sup_{t\in[0,T_0/\epsilon^2]} \|V(\cdot,\cdot,t)\|_{L^2} + C\epsilon^{5/2}. \tag{3.17}$$

Hence, to have (3.15) with α_0, $\beta_0 > \frac{1}{2}$, we only need

$$\sup_{t\in[0,T_0/\epsilon^2]} \|V(\cdot,\cdot,t)\|_{L^2} = \mathcal{O}\left(\epsilon^{\beta}\right), \qquad \sup_{t\in[0,T_0/\epsilon^2]} \|P(\cdot,\cdot,t)\|_{L^2} = \mathcal{O}\left(\epsilon^{\alpha}\right), \tag{3.18}$$

with α, $\beta > \frac{1}{2}$. Again, we intend to prove that the estimates can be closed with $\alpha = \beta = \frac{5}{2}$.

3.2.1. First energy level. While $P(x,z,t)$ can be controlled directly from Eq. (3.14), the estimate of $V(x,z,t)$ relies on *a priori* energy bounds. Multiplication of Eq. (3.13) by $V_t(x,z,t)$ and further integration by parts in (x,z) over $\mathbb{R}^2$ lead to

$$\frac{\mathrm{d}\mathcal{H}_1}{\mathrm{d}t} = \int_{\mathbb{R}^2} \left(\epsilon^4 |A|^2 V_t^2 + \frac{1}{2} N_t V_t^2 + \epsilon^2 \omega_0^2 (A)_{\omega_0} P V_t - 2i\omega_0 \epsilon^4 |A|^2 V V_t + \epsilon^8 R_8^{(V)} V_t\right) \mathrm{d}x\mathrm{d}z, \tag{3.19}$$

where we introduced the first energy functional

$$\mathcal{H}_1 := \frac{1}{2} \int_{\mathbb{R}^2} \left[\left(1 + \epsilon^2 M + N\right) V_t^2 + V_x^2 + V_z^2\right] \mathrm{d}x\mathrm{d}z. \tag{3.20}$$

This yields the estimate

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}_1}{\mathrm{d}t} \le\, & 2\epsilon^4 \|A\|_{L^\infty}^2 \mathcal{H}_1 + \|N_t\|_{L^\infty} \mathcal{H}_1 + 2\sqrt{2}\epsilon^2\omega_0^2 \|A\|_{L^\infty} \|P\|_{L^2} \mathcal{H}_1^{1/2} \\ & + 2\sqrt{2}\epsilon^4\omega_0 \|A\|_{L^\infty}^2 \|V\|_{L^2} \mathcal{H}_1^{1/2} + \sqrt{2}\epsilon^{13/2} \left\|R_8^{(V)}\right\|_{L^2_{X,Z}} \mathcal{H}_1^{1/2},\end{aligned} \tag{3.21}$$

where we recall that we are loosing $\epsilon^{3/2}$ when computing L^2-norms of the residual terms like $R_8^{(V)}$, because of integration in the slow variables $X = \epsilon x$ and $Z = \epsilon^2 z$.

Let $Q_1 := \mathcal{H}_1^{1/2}$ and assume that we can prove

$$\sup_{t\in[0,T_0/\epsilon^2]} Q_1 = \mathcal{O}\left(\epsilon^{\delta_1}\right), \tag{3.22}$$

for some $\delta_1 > 0$. Since $V|_{t=0} = 0$, Corollary A implies that for $t \in \left[0, T_0/\epsilon^2\right]$, we have

$$\|V\|_{L^2} \le \frac{T_0}{\epsilon^2} \sup_{t\in[0,T_0/\epsilon^2]} \|V_t\|_{L^2} \le \frac{\sqrt{2}T_0}{\epsilon^2} \sup_{t\in[0,T_0/\epsilon^2]} Q_1, \tag{3.23}$$

and hence $\sup_{t\in[0,T_0/\epsilon^2]} \|V\|_{L^2} = \mathcal{O}\left(\epsilon^{\delta_1-2}\right)$, that is, $\beta = \delta_1 - 2$ in (3.18).

Similarly, by the Gagliardo–Nirenberg inequality with $\sigma = 1$ (Proposition C), we estimate the nonlinear term in (3.14)

$$\|V\|_{L^4}^2 \le C_\sigma \|V\|_{L^2} \|\nabla V\|_{L^2}.$$

Since $P|_{t=0} = 0$, Corollary A implies that for $t \in \left[0, T_0/\epsilon^2\right]$, we have

$$\begin{aligned}\|P\|_{L^2} \le \frac{T_0}{\epsilon^2} \sup_{t\in[0,T_0/\epsilon^2]} \|P_t\|_{L^2} \le\, & \epsilon^{9/2} T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\|R_8^{(P)}\right\|_{L^2_{X,Z}} + 2C_\sigma T_0^2 \epsilon^{-4} \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1\right)^2 \\ & + 2\sqrt{2}T_0 \left(T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\|R_4^{(P)}\right\|_{L^\infty} + \frac{2}{\omega_0} \sup_{T\in[0,T_0]} \|A\|_{L^\infty}\right) \sup_{t\in[0,T_0/\epsilon^2]} Q_1,\end{aligned}$$

Hence, $\sup_{t\in[0,T_0/\epsilon^2]} \|P\|_{L^2} = \mathcal{O}\left(\epsilon^{9/2} + \epsilon^{\delta_1} + \epsilon^{2\delta_1-4}\right)$, that is, in (3.18),

$$\alpha = \min\left\{\frac{9}{2},\, \delta_1,\, 2\delta_1 - 4\right\}. \tag{3.24}$$

To control Q_1, we need to bound $\|N_t\|_{L^\infty}$ from Eq. (1.15),

$$\|N_t\|_{L^\infty} \leq 2\epsilon^4 \|A\|_{L^\infty}^2 + 4\epsilon^2 \|A\|_{L^\infty} \|U\|_{L^\infty} + \|U\|_{L^\infty}^2 , \tag{3.25}$$

where $\|U\|_{L^\infty}$ is controlled using (3.8) by

$$\|U\|_{L^\infty} \leq 2\epsilon^4 \|B\|_{L^\infty} + 2\epsilon^4 \|D\|_{L^\infty} + \|V\|_{L^\infty} . \tag{3.26}$$

By Sobolev's embedding (Proposition B), we can bound

$$\|V\|_{L^\infty} \leq C \|V\|_{H^2} , \tag{3.27}$$

if we assume the L^2-norm of second derivatives of V is controlled by some quantity Q_2 to be introduced later in Eq. (3.37), that is,

$$\|V_{xx}\|_{L^2} , \|V_{zz}\|_{L^2} \leq \sqrt{2} Q_2, \qquad \sup_{t\in[0,T_0/\epsilon^2]} Q_2 = \mathcal{O}(\epsilon^{\delta_2}), \tag{3.28}$$

for some $\delta_2 > 0$. Bound (3.23) and (3.28) imply that there is $C_0 > 0$ such that

$$\|V\|_{L^\infty} \leq C_0 \left[Q_2 + T_0\epsilon^{-2} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right]. \tag{3.29}$$

As we will see, $\sup_{t\in[0,T_0/\epsilon^2]} \|V\|_{L^\infty}$ is always bigger than $\mathcal{O}(\epsilon^4)$, so bound (3.26) implies that there is $C > 0$ such that

$$\|U\|_{L^\infty} \leq C \|V\|_{L^\infty} .$$

Then, bound (3.25) yields

$$\begin{aligned} \|N_t\|_{L^\infty} \leq{}& 2\epsilon^4 \|A\|_{L^\infty}^2 + 4C_0\epsilon^2 \|A\|_{L^\infty} Q_2 + C_0^2 Q_2^2 \\ &+\epsilon^{-4} C_0^2 T_0^2 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right)^2 + 2C_0 T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \left(\epsilon^{-2} C_0 Q_2 + 2\|A\|_{L^\infty}\right). \end{aligned} \tag{3.30}$$

Combining all together, Eq. (3.21) yields

$$\frac{\mathrm{d}Q_1}{\mathrm{d}t} \leq I_1 Q_1 + J_1,$$

where

$$\begin{aligned} I_1 ={}& 4\epsilon^4 \left(\sup_{T\in[0,T_0]} \|A\|_\infty \right)^2 + 2\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2 + \frac{C_0^2}{2} \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2 \right)^2 \\ &+\epsilon^{-2} C_0^2 T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2 + \frac{C_0^2 T_0^2}{2}\epsilon^{-4} \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right)^2 \\ &+2C_0 T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \end{aligned}$$

and

$$\begin{aligned} J_1 ={}& \sqrt{2}\epsilon^{13/2} \left(\frac{1}{2} \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_8^{(V)} \right\|_{L^2_{X,Z}} + \omega_0^2 T_0 \sup_{t\in[0,T_0/\epsilon^2]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_8^{(P)} \right\|_{L^2_{X,Z}} \right) \\ &+2\epsilon^2\omega_0 T_0 \left(\left(4+\sqrt{2}\right) \left(\sup_{T\in[0,T_0]} \|A\|_{L^\infty} \right)^2 + 2T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_4^{(P)} \right\|_{L^\infty} \right) \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \\ &+2\sqrt{2}\epsilon^{-2}\omega_0^2 T_0^2 C_\sigma \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1 \right)^2 . \end{aligned}$$

By Gronwall's inequality (Proposition D) with $Q_1|_{t=0} = 0$, we have, for $t \in \left[0, T_0/\epsilon^2\right]$,

$$Q_1 \leq T_0 \epsilon^{-2} J_1 e^{I_1 T_0 \epsilon^{-2}}. \tag{3.31}$$

To prevent divergence of the exponential factor $e^{I_1 T_0 \epsilon^{-2}}$ as $\epsilon \to 0$, we require that $I_1 \epsilon^{-2}$ be finite as $\epsilon \to 0$, that is,

$$\min\{\delta_2,\, 2\delta_2 - 2,\, \delta_2 + \delta_1 - 4,\, 2\delta_1 - 6,\, \delta_1 - 2\} \geq 0. \tag{3.32}$$

We also want $\delta_1 > 4$ so that the quadratic term $\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1\right)^2$ in $T_0\epsilon^{-2} J_1$ is negligible. Moreover, we require T_0 to be small enough such that

$$2\omega_0 T_0^2 e^{I_1 T_0 \epsilon^{-2}} \left(\left(4 + \sqrt{2}\right) \left(\sup_{t\in[0,T_0/\epsilon^2]} \|A\|_{L^\infty} \right)^2 + 2T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_4^{(P)} \right\|_{L^\infty} \right) < 1,$$

then, the coefficient of the linear term $\sup_{t\in[0,T_0/\epsilon^2]} Q_1$ in $T_0\epsilon^{-2} J_1$ is smaller than one.

With these constraints, we obtain from (3.31) that

$$\sup_{t\in[0,T_0/\epsilon^2]} Q_1 \leq C\epsilon^{9/2} T_0 e^{I_1 T_0 \epsilon^{-2}} \left(\sup_{t\in[0,T_0/\epsilon^2]} \left\| R_8^{(V)} \right\|_{L^2_{X,Z}} + 2\omega_0^2 T_0 \sup_{t\in[0,T_0/\epsilon^2]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \left\| R_8^{(P)} \right\|_{L^2_{X,Z}} \right),$$

hence $\sup_{t\in[0,T_0/\epsilon^2]} Q_1 = \mathcal{O}(\epsilon^{9/2})$. The conditions (3.24) and (3.32) imply that

$$\alpha = \delta_1 = \frac{9}{2}, \quad \beta = \delta_1 - 2 = \frac{5}{2}, \tag{3.33}$$

if we additionally require

$$\delta_2 \geq 1. \tag{3.34}$$

We will ensure that this constraint on δ_2 is satisfied by continuing next with *a priori* energy estimates on the second derivatives of V.

3.2.2. Second energy level. Acting on Eq. (3.13) with the operator $V_{xt}\partial_x + V_{zt}\partial_z + V_{tt}\partial_t$ and integrating in (x, z) over $\mathbb{R}^2$, we introduce the second energy functional:

$$\mathcal{H}_2 := \frac{1}{2} \int\limits_{\mathbb{R}^2} \left(\left(1 + \epsilon^2 M + N\right) V_{tt}^2 + \left(2 + \epsilon^2 M + N\right)\left(V_{xt}^2 + V_{zt}^2\right) + V_{xx}^2 + V_{zz}^2 + 2V_{xz}^2 \right) \mathrm{d}x\mathrm{d}z. \tag{3.35}$$

Long but straightforward computations show that the rate of change of the second energy functional is given by

$$\frac{\mathrm{d}\mathcal{H}_2}{\mathrm{d}t} = \sum_{n=1}^{9} K_n, \tag{3.36}$$

where

$$K_1 = \epsilon^4 \int\limits_{\mathbb{R}^2} |A|^2 \left(V_{tt}^2 + V_{tx}^2 + V_{tz}^2\right) \mathrm{d}x\mathrm{d}z,$$

$$K_2 = -\epsilon^3 \int\limits_{\mathbb{R}^2} V_{tt} \left(V_{tx} M_X + \epsilon V_{tz} M_Z + 2\epsilon |A|^2 V_{tt}\right) \mathrm{d}x\mathrm{d}z,$$

$$K_3 = \frac{1}{2} \int\limits_{\mathbb{R}^2} N_t \left(V_{tx}^2 + V_{tz}^2 - V_{tt}^2\right) \mathrm{d}x\mathrm{d}z,$$

$$K_4 = \int\limits_{\mathbb{R}^2} N \left[V_{ttx} V_{tx} + V_{ttz} V_{tz} + V_{tt} \left(V_{txx} + V_{tzz}\right)\right] \mathrm{d}x\mathrm{d}z,$$

$$K_5 = \epsilon^2 \omega_0^2 \int\limits_{\mathbb{R}^2} (A)_{\omega_0} \left(P_x V_{tx} + P_z V_{tz} + P_t V_{tt}\right) \mathrm{d}x\mathrm{d}z,$$

$$K_6 = \epsilon^2 \omega_0^2 \int\limits_{\mathbb{R}^2} P \left[\epsilon \left(A_X\right)_{\omega_0} V_{tx} + \left(\left(i\omega_0 A\right)_{\omega_0} + \epsilon^2 \left(A_Z\right)_{\omega_0}\right) V_{tz} + \left(-\left(i\omega_0 A\right)_{\omega_0} + \epsilon^2 \left(A_T\right)_{\omega_0}\right) V_{tt}\right] \mathrm{d}x\mathrm{d}z,$$

$$K_7 = -2i\epsilon^5 \omega_0 \int\limits_{\mathbb{R}^2} V \left(V_{tx} \partial_X |A|^2 + \epsilon V_{tz} \partial_Z |A|^2 + \epsilon V_{tt} \partial_T |A|^2\right) \mathrm{d}x\mathrm{d}z,$$

$$K_8 = -2i\epsilon^4 \omega_0 \int\limits_{\mathbb{R}^2} |A|^2 \left(V_{tx} V_x + V_{tz} V_z + V_{tt} V_t\right) \mathrm{d}x\mathrm{d}z,$$

$$K_9 = \epsilon^8 \int\limits_{\mathbb{R}^2} \left(\epsilon V_{tx} \partial_X R_8^{(V)} + V_{tz} \partial_z R_8^{(V)} + V_{tt} \partial_t R_8^{(V)}\right) \mathrm{d}x\mathrm{d}z.$$

We estimate the terms in (3.36) as follows:

$$|K_1| \le 2\epsilon^4 \|A\|_{L^\infty}^2 \mathcal{H}_2,$$

$$|K_2| \le 4\epsilon^3 \left(\left\|\nabla_{(X,Z)} M\right\|_{L^\infty} + \epsilon \|A\|_{L^\infty}^2\right) \mathcal{H}_2,$$

$$|K_3| \le \|N_t\|_{L^\infty} \mathcal{H}_2,$$

$$|K_4| \le \|N\|_{L^\infty} \left(\|V_{ttx}\|_{L^2} + \|V_{ttz}\|_{L^2} + \|V_{xxt}\|_{L^2} + \|V_{zzt}\|_{L^2}\right) \mathcal{H}_2^{1/2},$$

$$|K_5| \le 2\sqrt{2}\epsilon^2 \omega_0^2 \|A\|_{L^\infty} \left(2 \|\nabla P\|_{L^2} + \|P_t\|_{L^2}\right) \mathcal{H}_2^{1/2},$$

$$|K_6| \le 2\sqrt{2}\epsilon^2 \omega_0^2 \|P\|_{L^2} \left(\epsilon \|A_X\|_{L^\infty} + \epsilon^2 \|A_Z\|_{L^\infty} + \epsilon^2 \|A_T\|_{L^\infty} + 2\omega_0 \|A\|_{L^\infty}\right) \mathcal{H}_2^{1/2},$$

$$|K_7| \le 4\epsilon^5 \omega_0 \|A\|_{L^\infty} \|V\|_{L^2} \left(\|A_X\|_{L^\infty} + \epsilon \|A_Z\|_{L^\infty} + \epsilon \|A_T\|_{L^\infty}\right) \mathcal{H}_2^{1/2},$$

$$|K_8| \le 12\epsilon^4 \omega_0 \|A\|_{L^\infty}^2 \mathcal{H}_1^{1/2} \mathcal{H}_2^{1/2},$$

$$|K_9| \le \sqrt{2}\epsilon^{13/2} \left(\epsilon \left\|\partial_X R_8^{(V)}\right\|_{L^2_{X,Z}} + \left\|\partial_z R_8^{(V)}\right\|_{L^2_{X,Z}} + \left\|\partial_t R_8^{(V)}\right\|_{L^2_{X,Z}}\right) \mathcal{H}_2^{1/2}.$$

Let $Q_2 := \mathcal{H}_2^{1/2}$, and we want to ensure that

$$\sup_{t \in [T_0/\epsilon^2]} Q_2 = \mathcal{O}(\epsilon^{\delta_2}), \tag{3.37}$$

for some $\delta_2 \geq 1$ according to (3.34). To proceed further, we shall use the bounds

$$\begin{aligned}\sup_{T\in[0,T_0]} \left\|\nabla_{(X,Z)}M\right\|_{L^\infty} &\leq T_0 \sup_{T\in[0,T_0]} \left\|\nabla_{(X,Z)}\partial_T M\right\|_{L^\infty} \\ &\leq 4T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{T\in[0,T_0]} \left\|\nabla_{(X,Z)}A\right\|_{L^\infty}\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in[0,T_0/\epsilon^2]} \|\nabla P\|_{L^2} \leq 2\sqrt{2}T_0 &\left[T_0 \sup_{t\in[0,T_0/\epsilon^2]} \left\|\nabla R_4^{(P)}\right\|_{L^\infty} + C_0 T_0 \epsilon^{-4} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 + C_0\epsilon^{-2} \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right] \\ &\times \sup_{t\in[0,T_0/\epsilon^2]} Q_1 + \frac{4\sqrt{2}T_0}{\omega_0} \sup_{T\in[0,T_0]} \left\|\nabla_{(X,Z)}A\right\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2, \end{aligned} \tag{3.38}$$

where we dropped terms which are of higher order of smallness under assumptions $\nabla R_8^{(P)} \in L^2\left(\mathbb{R}^2\right),\ R_4^{(P)} \in L^\infty\left(\mathbb{R}^2\right),\ \nabla A \in L^\infty\left(\mathbb{R}^2\right)$.

To estimate the K_4 term, we control $\|N\|_{L^\infty}$ by using (3.30) and Corollary A:

$$\|N\|_{L^\infty} \leq \epsilon^{-2} T_0 \sup_{t\in[0,T_0/\epsilon^2]} \|N_t\|_{L^\infty}.$$

Additionally, we need to bound the third derivatives V_{ttx}, V_{ttz}, V_{xxt}, V_{zzt}, in which L^2-norms are controlled in terms of the quantity Q_3 that will be introduced later in Eq. (3.46), that is,

$$\|V_{ttx}\|_{L^2}, \|V_{ttz}\|_{L^2}, \|V_{xxt}\|_{L^2}, \|V_{zzt}\|_{L^2} \leq \sqrt{2}Q_3, \qquad \sup_{t\in[0,T_0/\epsilon^2]} Q_3 = \mathcal{O}(\epsilon^{\delta_3}), \tag{3.39}$$

for some $\delta_3 > 0$. Then,

$$\begin{aligned}|K_4| \leq 2T_0 &\left[2\epsilon^2 \left(\sup_{T\in[0,T_0]} \|A\|_{L^\infty}\right)^2 + 4C_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2 + \epsilon^{-2} C_0^2 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2\right. \\ &+\epsilon^{-6} C_0 T_0^2 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1\right)^2 + 2C_0^2 T_0 \epsilon^{-4} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2 \\ &\left.+ 4C_0 T_0 \epsilon^{-2} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{T\in[0,T_0]} \|A\|_{L^\infty}\right] \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_3\right) Q_2.\end{aligned}$$

Details for other K-terms in (3.36) can be elaborated using the previous bounds. Combining all together and neglecting *a priori* smaller source terms in K_9 in comparison with other terms, we obtain from Eq. (3.36):

$$\frac{\mathrm{d}Q_2}{\mathrm{d}t} \leq I_2 Q_2 + J_2^{(1)} + J_2^{(2)} \sup_{t\in[0,T_0/\epsilon^2]} Q_3, \tag{3.40}$$

where

$$I_2 = 4\epsilon^3 T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \|\nabla A\|_{L^\infty} + 2\epsilon^2 C_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2 + \frac{C_0^2}{2}\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2$$
$$+\frac{\epsilon^{-4}C_0^2T_0^2}{2}\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1\right)^2 + 2\epsilon^{-2}C_0T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_1$$
$$+\epsilon^{-4}C_0^2T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2,$$
$$J_2^{(1)} = 4\omega_0^2 T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty}\left[2\sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_1 + \sqrt{2}C_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right.$$
$$\left.+\epsilon^{-2}\frac{T_0}{\sqrt{2}}(2C_0 + C_\sigma\omega_0)\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1\right)^2 + \frac{2}{\omega_0}\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right],$$

and

$$J_2^{(2)} = T_0\left[2\epsilon^2\left(\sup_{T\in[0,T_0]} \|A\|_{L^\infty}\right)^2 + 4C_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2 + \epsilon^{-2}C_0^2\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2\right.$$
$$+\epsilon^{-6}C_0T_0^2\left(\sup_{t\in[0,T_0/\epsilon^2]} Q_1\right)^2 + 2\epsilon^{-4}C_0^2T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2$$
$$\left.+4\epsilon^{-2}C_0T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_1\right].$$

By Gronwall's inequality (Proposition D) with $Q_2|_{t=0} = 0$, we obtain

$$Q_2 \le T_0\epsilon^{-2}J_2 e^{I_2T_0\epsilon^{-2}}, \quad J_2 = J_2^{(1)} + J_2^{(2)} \sup_{t\in[0,T_0/\epsilon^2]} Q_3.$$

To bound the exponential factor as $\epsilon \to 0$, we require $I_2\epsilon^{-2}$ to be finite as $\epsilon \to 0$, that is,

$$\min\{\delta_2,\ 2\delta_2 - 2,\ 2\delta_1 - 6,\ \delta_1 + \delta_2 - 6,\ \delta_1 - 4\} \ge 0. \tag{3.41}$$

Taking into account (3.33), this constraint reduces to the condition $\delta_2 \ge \frac{3}{2}$. On the other hand, the source term in $T_0\epsilon^{-2}J_2^{(1)}$ yields $\sup_{t\in[0,T_0/\epsilon^2]} Q_2 = \mathcal{O}(\epsilon^{5/2})$, that is,

$$\delta_2 = \frac{5}{2}. \tag{3.42}$$

Under this condition, the linear and quadratic terms with respect to $\sup_{t\in[0,T_0/\epsilon^2]} Q_2$ are sufficiently small in $T_0\epsilon^{-2}J_2$, if T_0 is sufficiently small and if we additionally require

$$\delta_3 \ge \delta_2. \tag{3.43}$$

Now, we proceed with *a priori* energy estimates on the next level to justify the constraint (3.43).

3.2.3. Third energy level. Acting on Eq. (3.13) with the operator $V_{xxt}\partial_x^2 + V_{zzt}\partial_z^2 + V_{ttt}\partial_t^2$ and integrating in (x,z) over $\mathbb{R}^2$, we introduce the third energy functional:

$$\mathcal{H}_3 := \frac{1}{2}\int_{\mathbb{R}^2}\left((1+\epsilon^2M+N)\left(V_{xxt}^2 + V_{zzt}^2 + V_{ttt}^2\right) + \frac{1}{2}\left(V_{xxx}^2 + V_{zzz}^2\right) + V_{xxz}^2 + V_{xzz}^2\right.$$
$$\left.+\frac{1}{2}[V_{xxx} - 2N_xV_{tt}]^2 + \frac{1}{2}[V_{zzz} - 2N_zV_{tt}]^2\right) \mathrm{d}x\mathrm{d}z. \tag{3.44}$$

Long but straightforward computations show that the rate of change of the third energy functional is given by

$$\frac{\mathrm{d}\mathcal{H}_3}{\mathrm{d}t} = \sum_{n=1}^{15} L_n, \tag{3.45}$$

where

$$L_1 = \epsilon^4 \int_{\mathbb{R}^2} |A|^2 \left(V_{xxt}^2 + V_{zzt}^2 - V_{ttt}^2\right) \mathrm{d}x\mathrm{d}z,$$

$$L_2 = \frac{1}{2} \int_{\mathbb{R}^2} N_t \left(V_{xxt}^2 + V_{zzt}^2 - 3V_{ttt}^2\right) \mathrm{d}x\mathrm{d}z,$$

$$L_3 = -\epsilon^4 \int_{\mathbb{R}^2} V_{tt} \left(V_{xxt} M_{XX} + \epsilon^2 V_{zzt} M_{ZZ} + \epsilon^2 V_{ttt} \partial_T |A|^2\right) \mathrm{d}x\mathrm{d}z,$$

$$L_4 = -2\epsilon^3 \int_{\mathbb{R}^2} \left(V_{xxt} V_{xtt} M_X + \epsilon V_{zzt} V_{ztt} M_Z\right) \mathrm{d}x\mathrm{d}z,$$

$$L_5 = -\int_{\mathbb{R}^2} \left[N_x \left(2V_{xxt} V_{xtt} + V_{xxx} V_{ttt} - V_{xtt}^2\right) + N_z \left(2V_{zzt} V_{ztt} + V_{zzz} V_{ttt} - V_{ztt}^2\right)\right] \mathrm{d}x\mathrm{d}z,$$

$$L_6 = \int_{\mathbb{R}^2} \left[V_{tt} \left(N_{xt} V_{xxx} + N_{zt} V_{zzz}\right) + V_{tt}^2 \left(N_{tt} + 2N_x N_{xt} + 2N_z N_{zt}\right)\right] \mathrm{d}x\mathrm{d}z,$$

$$L_7 = 2 \int_{\mathbb{R}^2} V_{tt} V_{ttt} \left(N_x^2 + N_z^2\right) \mathrm{d}x\mathrm{d}z,$$

$$L_8 = \epsilon^2 \omega_0^2 \int_{\mathbb{R}^2} (A)_{\omega_0} \left(P_{xx} V_{xxt} + P_{zz} V_{zzt} + P_{tt} V_{ttt}\right) \mathrm{d}x\mathrm{d}z,$$

$$L_9 = 2\epsilon^3 \int_{\mathbb{R}^2} \left(i\omega_0^3 A\right)_{\omega_0} \left(P_z V_{zzz} - P_t V_{ttt}\right) \mathrm{d}x\mathrm{d}z,$$

$$L_{10} = 2\epsilon^2 \omega_0^2 \int_{\mathbb{R}^2} \left((A_X)_{\omega_0} P_x V_{xxt} + \epsilon (A_Z)_{\omega_0} P_z V_{zzt} + \epsilon (A_T)_{\omega_0} P_t V_{ttt}\right) \mathrm{d}x\mathrm{d}z,$$

$$L_{11} = \epsilon^4 \omega_0^2 \int_{\mathbb{R}^2} P \left((A_{XX})_{\omega_0} V_{xxt} + \epsilon^2 (A_{ZZ})_{\omega_0} V_{zzt} + \epsilon^2 (A_{TT})_{\omega_0} V_{ttt}\right) \mathrm{d}x\mathrm{d}z,$$

$$L_{12} = -2i\epsilon^6 \omega_0 \int_{\mathbb{R}^2} V \left(V_{xxt} \partial_X^2 |A|^2 + \epsilon^2 V_{zzt} \partial_Z^2 |A|^2 + \epsilon^2 V_{ttt} \partial_T^2 |A|^2\right) \mathrm{d}x\mathrm{d}z,$$

$$L_{13} = -4i\epsilon^5 \omega_0 \int_{\mathbb{R}^2} \left(V_x V_{xxt} \partial_X |A|^2 + \epsilon V_z V_{zzt} \partial_Z |A|^2 + \epsilon V_t V_{ttt} \partial_T |A|^2\right) \mathrm{d}x\mathrm{d}z,$$

$$L_{14} = -2i\epsilon^4 \omega_0 \int_{\mathbb{R}^2} |A|^2 \left(V_{xx} V_{xxt} + V_{zz} V_{zzt} + V_{tt} V_{ttt}\right) \mathrm{d}x\mathrm{d}z,$$

$$L_{15} = \epsilon^8 \int_{\mathbb{R}^2} \left(\epsilon^2 V_{xxt} \partial_X^2 R_8^{(V)} + V_{zzt} \partial_z^2 R_8^{(V)} + V_{ttt} \partial_t^2 R_8^{(V)}\right) \mathrm{d}x\mathrm{d}z.$$

We shall estimate these terms as follows:

$$
\begin{aligned}
|L_1| &\leq 2\epsilon^4 \|A\|^2_{L^\infty} \mathcal{H}_3,\\
|L_2| &\leq \|N_t\|_{L^\infty} \mathcal{H}_3,\\
|L_3| &\leq 2\epsilon^4 \left(\|M_{XX}\|_{L^\infty} + \epsilon^2 \|M_{ZZ}\|_{L^\infty} + 2\epsilon^2 \|A\|_{L^\infty} \|A_T\|_{L^\infty}\right) \mathcal{H}_3,\\
|L_4| &\leq 4\epsilon^3 \left(\|M_X\|_{L^\infty} + \epsilon \|M_Z\|_{L^\infty}\right) \mathcal{H}_3,\\
|L_5| &\leq 12 \|\nabla N\|_{L^\infty} \mathcal{H}_3,\\
|L_6| &\leq 4 \|\nabla N_t\|_{L^\infty} \mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2} + 2\left(\|N_{tt}\|_{L^\infty} + 8\|\nabla N\|_{L^\infty} \|\nabla N_t\|_{L^\infty}\right) \mathcal{H}_2,\\
|L_7| &\leq 16 \|\nabla N\|^2_{L^\infty} \mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2},\\
|L_8| &\leq 2\sqrt{2}\epsilon^2\omega_0^2 \|A\|_{L^\infty} \left(\|\Delta P\|_{L^2} + \|P_{tt}\|_{L^2}\right) \mathcal{H}_3^{1/2},\\
|L_9| &\leq 4\sqrt{2}\epsilon^3\omega_0^3 \|A\|_{L^\infty} \left(\|\nabla P\|_{L^2} + \|P_t\|_{L^2}\right) \mathcal{H}_3^{1/2},\\
|L_{10}| &\leq 4\sqrt{2}\epsilon^2\omega_0^2 \left[\left(\|A_X\|_{L^\infty} + \epsilon \|A_Z\|_{L^\infty}\right) \|\nabla P\|_{L^2} + \epsilon \|A_T\|_{L^\infty} \|P_t\|_{L^2}\right] \mathcal{H}_3^{1/2},\\
|L_{11}| &\leq 2\sqrt{2}\epsilon^4\omega_0^2 \left(\|A_{XX}\|_{L^\infty} + \epsilon^2 \|A_{ZZ}\|_{L^\infty} + \epsilon^2 \|A_{TT}\|_{L^\infty}\right) \mathcal{H}_3^{1/2},\\
|L_{12}| &\leq 4\sqrt{2}\epsilon^6\omega_0 \|V\|_{L^2} \Big(\|A\|_{L^\infty} \left[\|A_{XX}\|_{L^\infty} + \epsilon^2 \|A_{ZZ}\|_{L^\infty} + \epsilon^2 \|A_{TT}\|_{L^\infty}\right]\\
&\quad + \|A_X\|^2_{L^\infty} + \epsilon^2 \|A_Z\|^2_{L^\infty} + \epsilon^2 \|A_T\|^2_{L^\infty}\Big) \mathcal{H}_3^{1/2},\\
|L_{13}| &\leq 32\epsilon^5\omega_0 \|A\|_{L^\infty} \left(\|A_X\|_{L^\infty} + \epsilon \|A_Z\|_{L^\infty} + \epsilon \|A_T\|_{L^\infty}\right) \mathcal{H}_1^{1/2}\mathcal{H}_3^{1/2},\\
|L_{14}| &\leq 12\epsilon^4\omega_0 \|A\|^2_{L^\infty} \mathcal{H}_2^{1/2}\mathcal{H}_3^{1/2},\\
|L_{15}| &\leq \sqrt{2}\epsilon^{13/2} \left(\epsilon^2 \left\|\partial_X^2 R_8^{(V)}\right\|_{L^2} + \left\|\partial_z^2 R_8^{(V)}\right\|_{L^2} + \left\|\partial_t^2 R_8^{(V)}\right\|_{L^2}\right) \mathcal{H}_3^{1/2}.
\end{aligned}
$$

Let $Q_3 := \mathcal{H}_3^{1/2}$, and we want to ensure that

$$\sup_{t\in[T_0/\epsilon^2]} Q_3 = \mathcal{O}(\epsilon^{\delta_3}), \tag{3.46}$$

for $\delta_3 \geq \delta_2 = \frac{5}{2}$. At this energy level, there will be no restriction on the upper bound of the time interval; therefore, we do not necessarily need to keep track of particular expressions of all the L-terms estimates, instead we will be looking at their order of smallness only.

To control the right-hand side of Eq. (3.45), we use (3.38), Corollary A, and Propositions B and C to obtain the following estimates

$$
\begin{aligned}
\sup_{t\in[0,T_0/\epsilon^2]} \|\nabla N\|_{L^\infty} &\leq 2T_0\epsilon^{-2} \sup_{t\in[0,T_0/\epsilon^2]} \Bigg[\epsilon^4 \sup_{T\in[0,T_0]} \|A\|^2_{L^\infty} + 2\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \|U\|_{L^\infty}\\
&\quad + \|\nabla U\|^2_{L^\infty} + \|U\|_{L^\infty} \|\nabla U\|_{L^\infty}\Bigg],\\
\sup_{t\in[0,T_0/\epsilon^2]} \|\Delta P\|_{L^2} &\leq T_0\epsilon^{-2} \sup_{t\in[0,T_0/\epsilon^2]} \Bigg[\epsilon^{13/2} \left\|\Delta R_8^{(P)}\right\|_{L^2_{X,Z}} + 2\epsilon^4 \left\|\Delta R_4^{(P)}\right\|_{L^\infty} \|V\|_{L^2}\\
&\quad +2\epsilon^4 \left\|R_4^{(P)}\right\|_{L^\infty} \|\Delta V\|_{L^2} + 4\epsilon^2\omega_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \|V_t\|_{L^2} + 4\frac{\epsilon^2}{\omega_0} \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \|\Delta V_t\|_{L^2}\\
&\quad +2\|V\|_{L^\infty} \|\Delta V\|_{L^2} + 2C_\sigma \|\nabla V\|_{L^2} \|\Delta V\|_{L^2}\Bigg],\\
\sup_{t\in[0,T_0/\epsilon^2]} \|P_{tt}\|_{L^2} &\leq \epsilon^{13/2} \sup_{t\in[0,T_0/\epsilon^2]} \left\|\Delta R_8^{(P)}\right\|_{L^2_{X,Z}} + 2\epsilon^4 \sup_{t\in[0,T_0/\epsilon^2]} \left\|R_4^{(P)}\right\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|V_t\|_{L^2}
\end{aligned}
$$

$$+2\epsilon^4 \sup_{t\in[0,T_0/\epsilon^2]} \left\|\partial_t R_4^{(P)}\right\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|V\|_{L^2} + 4\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|V_t\|_{L^2}$$
$$+4\frac{\epsilon^2}{\omega_0} \sup_{t\in[0,T_0/\epsilon^2]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|V_{tt}\|_{L^2} + 2 \sup_{t\in[0,T_0/\epsilon^2]} \|V\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|V_t\|_{L^2},$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|\nabla N_t\|_{L^\infty} \le 4\epsilon^4\omega_0 \left(\sup_{T\in[0,T_0]} \|A\|_{L^\infty}\right)^2 + 4\epsilon^2\omega_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|U\|_{L^\infty}$$
$$+4\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla U\|_{L^\infty} + 2 \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla U\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|U\|_{L^\infty},$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|N_{tt}\|_{L^\infty} \le 4\epsilon^4\omega_0 \left(\sup_{T\in[0,T_0]} \|A\|_{L^\infty}\right)^2 + 4\epsilon^2\omega_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|U\|_{L^\infty}$$
$$+4\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla U\|_{L^\infty} + 2 \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla U\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} \|U\|_{L^\infty},$$

and

$$\sup_{T\in[0,T_0]} \left\|\Delta_{(X,Z)} M\right\|_{L^\infty} \le 4T_0 \left[\left(\sup_{T\in[0,T_0]} \|\nabla A\|_{L^\infty}\right)^2 + \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \sup_{T\in[0,T_0]} \|\Delta A\|_{L^\infty}\right],$$

where smaller terms are neglected under assumption ΔA, ∇A, $\partial_T A \in L^\infty\left(\mathbb{R}^2\right)$.

Taking into account (3.33), (3.42) and (3.43), we can drop *a priori* smaller terms and hence obtain

$$\frac{\mathrm{d}\mathcal{H}_3}{\mathrm{d}t} \le 24T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left[\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} + 2 \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right] \mathcal{H}_3$$
$$+8\omega_0 T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left[\epsilon^{-2} C_0\omega_0 T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right.$$
$$\left. + C_0\omega_0 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2 + 4\epsilon^2 \sup_{T\in[0,T_0]} \|\nabla A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right] \mathcal{H}_3^{1/2}$$
$$+8\epsilon^2\omega_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2 \left(\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} + C_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right). \tag{3.47}$$

The source term can be dropped if

$$\min\{4+2\delta_2,\, 2+3\delta_2\} > \max\{2\delta_3+\delta_2, \delta_3+2\delta_2\}, \tag{3.48}$$

hence, $\delta_3 < \frac{13}{4}$. Neglecting the source term in (3.47), we obtain

$$\frac{\mathrm{d}Q_3}{\mathrm{d}t} \le I_3 Q_3 + J_3, \tag{3.49}$$

where

$$I_3 = 12T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left[\epsilon^2 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} + 2 \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right],$$

and

$$J_3 = 4\omega_0 T_0 \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left[\epsilon^{-2} C_0 \omega_0 T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2 + C_0\omega_0 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2 + 4\epsilon^2 \sup_{T\in[0,T_0]} \|\nabla A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right].$$

By Gronwall's inequality (Proposition D) with $Q_3|_{t=0} = 0$, we obtain

$$\sup_{t\in[0,T_0/\epsilon^2]} Q_3 \leq 4\epsilon^{-2}\omega_0^2 T_0^2 e^{I_3 T_0 \epsilon^{-2}} \sup_{T\in[0,T_0]} \|A\|_{L^\infty} \left[\epsilon^{-2} C_0 T_0 \sup_{t\in[0,T_0/\epsilon^2]} Q_1 \sup_{t\in[0,T_0/\epsilon^2]} Q_2 + C_0 \left(\sup_{t\in[0,T_0/\epsilon^2]} Q_2\right)^2 + 4\epsilon^2 \sup_{T\in[0,T_0]} \|\nabla A\|_{L^\infty} \sup_{t\in[0,T_0/\epsilon^2]} Q_2\right].$$

Taking into account (3.33), (3.42), and (3.43), we deduce that $\sup_{t\in[0,T_0/\epsilon^2]} Q_3 = \mathcal{O}(\epsilon^{5/2})$, that is,

$$\delta_3 = \frac{5}{2}, \tag{3.50}$$

which is compatible with the condition $\delta_3 < \frac{13}{4}$ that follows from the inequality (3.48). Hence, the third energy level is controlled, and all *a priori* energy estimates are closed.

3.3. Proof of Theorem 1

According to (3.33),(3.42), and (3.50), we have the following estimates

$$\sup_{t\in[0,T_0/\epsilon^2]} \|V\|_{L^2} = \mathcal{O}(\epsilon^{5/2}), \qquad \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla V\|_{L^2} = \mathcal{O}(\epsilon^{9/2}), \tag{3.51}$$

$$\sup_{t\in[0,T_0/\epsilon^2]} \|\Delta V\|_{L^2} = \mathcal{O}(\epsilon^{5/2}), \qquad \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla\Delta V\|_{L^2} = \mathcal{O}(\epsilon^{5/2}), \tag{3.52}$$

and

$$\sup_{t\in[0,T_0/\epsilon^2]} \|P\|_{L^2} = \mathcal{O}(\epsilon^{9/2}), \qquad \sup_{t\in[0,T_0/\epsilon^2]} \|\nabla P\|_{L^2} = \mathcal{O}(\epsilon^{5/2}), \qquad \sup_{t\in[0,T_0/\epsilon^2]} \|\Delta P\|_{L^2} = \mathcal{O}(\epsilon^{5/2}). \tag{3.53}$$

From (3.16), (3.17), and the previous bounds, we obtain bounds (3.15) with $\alpha_0 = \beta_0 = \frac{5}{2}$. Combining these bounds with similar bounds for the L^2-norms of derivatives of V and P, we obtain the bounds (1.19) of Theorem 1.

Note that, the estimates involving $R_8^{(V)}$, $R_8^{(P)}$, $R_6^{(P)}$, and $R_4^{(P)}$ rely on the smoothness of $A(X,Z,T)$, $B(X,Z,T)$, and $D(X,Z,T)$. This smoothness is gained with the use of Lemma 3 and Remark 2 provided that the initial data $A_0\left(X,Z\right)$ are sufficiently smooth. Indeed, the most stringent requirements come from the estimates performed on the third energy level, where we have imposed conditions $\partial_X^2 A$, $\partial_Z^2 A$, $\partial_T^2 A \in L^\infty\left(\mathbb{R}^2\right)$ for all $T \in [0,T_0]$ and $\partial_X^2 R_8^{(V)}$, $\partial_Z^2 R_8^{(V)}$, $\partial_T^2 R_8^{(V)} \in L^2\left(\mathbb{R}^2\right)$ for all $t \in \left[0, T_0/\epsilon^2\right]$. Expressing T-derivatives from Eqs. (2.23), (3.10) and (3.11) and differentiating one more time with respect to T, we can see that these requirements are satisfied if $A_0 \in H^8\left(\mathbb{R}^2\right)$.

We have already obtained a bound for $\sup_{t\in[0,T_0/\epsilon^2]} \|U\|_{L^\infty}$ when performing estimates on the first energy level. Similarly, applying Sobolev's embeddings (Proposition B) to the derivatives of (3.8) and using (3.51)–(3.52), we control

$$\sup_{t\in[0,T_0/\epsilon^2]}\|U\|_{L^\infty}=\mathcal{O}(\epsilon^{5/2}),\qquad \sup_{t\in[0,T_0/\epsilon^2]}\|\nabla U\|_{L^\infty}=\mathcal{O}(\epsilon^{5/2}),\qquad \sup_{t\in[0,T_0/\epsilon^2]}\|U_t\|_{L^\infty}=\mathcal{O}(\epsilon^{5/2}). \tag{3.54}$$

This allows us to apply Lemma 2 to extend validity of local solution E with small initial data up to the time $t_0 = T_0/\epsilon^2$ while preserving the accuracy of the error bounds (1.19). The proof of Theorem 1 is complete.

Appendix A: Elements of functional analysis

For a positive integer s, we denote the L^2-based Sobolev space by $H^s(\mathbb{R}^2) := W^{s,2}(\mathbb{R}^2)$ and endow it with the norm:

$$\|f\|_{H^s} := \left(\sum_{k+l=s}\int_{\mathbb{R}^2}\left|\partial_x^k\partial_z^l f\right|^2 \mathrm{d}x\mathrm{d}z\right)^{1/2} + \left(\int_{\mathbb{R}^2}|f|^2\,\mathrm{d}x\mathrm{d}z\right)^{1/2}.$$

For any $p \geq 1$, Lebesgue spaces $L^p(\mathbb{R}^2)$ are endowed with the norm

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^2}|f(x,z)|^p\,\mathrm{d}x\mathrm{d}z\right)^{1/p}.$$

The L^∞ space is obtained in the limiting procedure:

$$\|f\|_{L^\infty} := \lim_{p\to\infty}\|f\|_{L^p} = \operatorname*{ess\,sup}_{(x,z)\in\mathbb{R}^2}|f(x,z)|\,.$$

Now, we assume that functions f in $H^s(\mathbb{R}^2)$ depend on an additional variable $t\in\mathbb{R}_+$. We will often write $f\in H^s(\mathbb{R}^2)$ implying $f(\cdot,\cdot,t)\in H^s(\mathbb{R}^2)$ for fixed t.

Lemma A. *Assume that f, $\partial_t f \in L^p(\mathbb{R}^2)$ and $\|f\|_{L^p}\neq 0$. Then for any $1\leq p<\infty$, we have*

$$\partial_t\|f\|_{L^p} \leq \|\partial_t f\|_{L^p}\,. \tag{A.1}$$

Proof. Clearly,

$$\partial_t\|f\|_{L^p}^p = p\,\|f\|_{L^p}^{p-1}\,\partial_t\|f\|_{L^p}\,. \tag{A.2}$$

On the other hand, for $1\leq p<\infty$, Lebesgue's dominated convergence theorem (valid since $f\in L^p(\mathbb{R}^2)$, $\partial_t f\in L^p(\mathbb{R}^2)$) ensures that differentiation can be performed under the integral sign which is then followed by an application of Hölder's inequality

$$\partial_t\|f\|_{L^p}^p = p\int_{\mathbb{R}^2}|f(x,z,t)|^{p-1}\,\partial_t f(x,z,t)\,\mathrm{d}x\mathrm{d}z \leq p\left\|f^{p-1}\right\|_{L^{p/(p-1)}}\|\partial_t f\|_{L^p} = p\,\|f\|_{L^p}^{p-1}\|\partial_t f\|_{L^p}\,. \tag{A.3}$$

Comparison of (A.2) and (A.3) furnishes the result (A.1). □

Corollary A. *Assume that f, $\partial_t f\in L^p(\mathbb{R}^2)$ for all $t\in[0,t_0]$ and some $p\geq 1$. Then, we have*

$$\|f\|_{L^p} \leq t_0\sup_{t\in[0,t_0]}\|\partial_t f\|_{L^p} + \left(\|f\|_{L^p}\right)\big|_{t=0}\,,\quad t\in[0,t_0]\,. \tag{A.4}$$

Proof. For $p=\infty$, the result follows from the fundamental theorem of calculus and integral Minkowski's inequality

$$\|f\|_{L^\infty} \leq \left\| \int_0^t \partial_\tau f \, \mathrm{d}\tau \right\|_{L^\infty} + \left(\|f\|_{L^\infty}\right)\big|_{t=0}$$
$$\leq t_0 \sup_{t\in[0,t_0]} \|\partial_t f\|_{L^\infty} + \left(\|f\|_{L^\infty}\right)\big|_{t=0}, \quad t \in [0, t_0].$$

For $p < \infty$, this result follows directly from Lemma A. □

Corollary B. *Let f, $\partial_t f \in H^s(\mathbb{R}^2)$ for all $t \in [0, t_0]$ and some $s \geq 0$. Then, we have*

$$\|f\|_{H^s} \leq t_0 \sup_{t\in[0,t_0]} \|\partial_t f\|_{H^s} + \left(\|f\|_{H^s}\right)\big|_{t=0}, \quad t \in [0, t_0]. \tag{A.5}$$

Proof. By Plancherel's theorem, we can employ the estimate (A.4) for $p = 2$ on the Fourier transform side $f(\boldsymbol{\xi}) \in L^2(\mathbb{R}^2)$

$$\|f\|_{H^s} = \left\| \left(1 + |\boldsymbol{\xi}|^2\right)^{s/2} \hat{f} \right\|_{L^2}$$
$$\leq t_0 \sup_{t\in[0,t_0]} \left\| \left(1 + |\boldsymbol{\xi}|^2\right)^{s/2} \partial_t \hat{f} \right\|_{L^2} + \left(\left\| \left(1 + |\boldsymbol{\xi}|^2\right)^{s/2} \hat{f} \right\|_{L^2} \right)\Bigg|_{t=0}.$$

Using Plancherel's theorem again, we obtain (A.5). □

We shall now list useful results: Banach algebra property, Sobolev embedding theorem, Gagliardo–Nirenberg inequality and Gronwall's inequality, and Banach fixed-point theorem. For the proofs, see [1] and Appendix B in [16].

Proposition A. (Banach algebra property) *For any $s > 1$, $H^s(\mathbb{R}^2)$ is a Banach algebra with respect to multiplication, that is, if f, $g \in H^s(\mathbb{R}^2)$, then there is a constant $C_s > 0$ (depending only on index s) such that*

$$\|fg\|_{H^s} \leq C_s \|f\|_{H^s} \|g\|_{H^s}. \tag{A.6}$$

Proposition B. (Sobolev embedding) *Assume that $f \in H^s(\mathbb{R}^2)$ for $s \geq 2$. Then, the function f is continuous on $\mathbb{R}^2$ decaying at infinity, and there is a constant $C_s > 0$ such that*

$$\|f\|_{L^\infty} \leq C_s \|f\|_{H^s}. \tag{A.7}$$

Proposition C. (Gagliardo–Nirenberg inequality) *Let $f \in H^1(\mathbb{R}^2)$. Then, for any $\sigma \geq 0$, there exists a constant $C_\sigma > 0$ such that*

$$\|f\|_{L^{2(\sigma+1)}}^{2(\sigma+1)} \leq C_\sigma \|\nabla f\|_{L^2}^{2\sigma} \|f\|_{L^2}^2. \tag{A.8}$$

Proposition D. (Gronwall's inequality) *Assume $g(t) \in C^1([0, t_0])$ satisfies*

$$\frac{dg(t)}{\mathrm{d}t} \leq ag(t) + b, \quad t \in [0, t_0].$$

for some constants a, $b > 0$ and $g(0) > 0$. Then, we have

$$g(t) \leq (g(0) + bt_0)\, e^{at}, \quad t \in [0, t_0]. \tag{A.9}$$

Proposition E. (Banach fixed-point theorem) *Let $\mathcal{B}$ be a closed nonempty set of the Banach space X, and let $K : \mathcal{B} \mapsto \mathcal{B}$ be a contraction operator, that is, for any x, $y \in \mathcal{B}$, there exists $0 \leq q < 1$ such that $\|K(x) - K(y)\|_X \leq q \|x - y\|_X$. Then, there exists a unique fixed point of K in $\mathcal{B}$, in other words, there exists a unique solution $x_0 \in \mathcal{B}$ such that $K(x_0) = x_0$.*

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