Chapter 1
Solution of a homogeneous version of Love type integral equation in different asymptotic regimes

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1.1 Introduction

For h, $a > 0$, we consider the following homogeneous Fredholm integral equation of the second kind

$$\frac{h}{\pi}\int_{-a}^{a}\frac{f(t)}{(x-t)^2+h^2}dt = \lambda f(x), \quad x \in (-a,a), \tag{1.1}$$

which can be viewed as a problem of finding eigenfunctions of the integral operator $P_h\chi_A\colon L^2(A) \to L^2(A)$ with

$$P_h[f](x) := (p_h \star f)(x) = \frac{h}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(x-t)^2+h^2}dt, \tag{1.2}$$

$$p_h(x) := \frac{h}{\pi}\frac{1}{x^2+h^2}, \tag{1.3}$$

and χ_A being the characteristic function of the interval $A := (-a,a)$.

Integral equations with kernel function (1.3) have a long history starting with [Sno], as the earliest mention we could trace, and up to very recent papers [TW1, TW2, TW3, Pro]. It most commonly arises in rotationally symmetric electrostatic

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[Lov] or fluid dynamics problems [Hut1] (in such contexts it is most famously known as Love equation), quantum-mechanical statistics of Fermi / Bose gases (known there as Gaudin / Lieb-Liniger equation, respectively), antiferromagnetic one-dimensional Heisenberg chains [Gri], and is relevant as well in other contexts such as probability theory [KP] and radiative transfer [vT]. Since p_h is the two-dimensional Poisson kernel for the upper half-plane, equation (1.1) has also applications to problems of approximation by harmonic functions [LP] and it is instrumental in some inverse source problems for the Poisson PDE (so-called inverse magnetization problems, see [BHLSW]).

The class of exactly solvable convolution integral equations on interval is extremely narrow and rarely exceds the class of equations with kernels whose Fourier transform is a rational function. Such approaches usually hinge on a matrix Wiener-Hopf factorization which is inapplicable due to non-smooth and non-algebraic (at infinity) behavior of the Fourier transform of the kernel function (1.3): $\hat{p}_h(k) = e^{-2\pi h|k|}$. Therefore, the main hope for an analytical solution is a structural approach (i.e. when exact solutions are determined up to constants that cannot be expressed in a closed-form) or an asymptotic one. Despite seeming simplicity of the kernel function p_h, the integral equation (1.1) evades applicability of relevant constructive techniques: both for exact structural [LM] and asymptotic solutions [KK, Hut2]. The problem of failure of asymptotic approaches (when the length of the interval is large) in [KK, Hut2] is the lack of sufficient decay of the kernel function at infinity (alternatively, the lack of existence of second-order derivative at $k = 0$ of $\hat{p}_h(k)$). The powerful approach of Leonard and Mullikin [LM] aiming to obtain essentially exact sine/cosine solutions (with frequencies to be determined from auxiliary equations which are unsolvable explicitly) breaks down since the inverse Laplace transform of the kernel function is not of constant sign which the authors claim to be merely a technical problem (according to them, the assumption of constant sign is made to "simplify the discussion"). However, from results of our approach we will see that change of the sign of this function that occurs infinite number of times results in a qualitatively different form of solutions than that for a large class of standard kernels. Such solutions being beyond simple trigonometric functions shifted by constants (see Figures of Section 1.5) are still reminiscent of solutions of a Sturm-Liouville problem.

To the best of our knowledge, the only available result in the literature regarding the equation (1.1) (except for its non-homogeneous version with $\lambda = \pm 1$) is the exact exponential decay law of eigenvalues [Wid] and a relevant reduction to a hypersingular equation which "appears too difficult to solve explicitly" [KK] (see also Section 1.6).

Consideration of the homogeneous version of the equation with the kernel (1.3) is the most general in a sense that the obtained solutions (eigenfunctions) permit solving a general non-homogeneous equation with λ different from an eigenvalue. Indeed, Mercer's theorem for positive definite kernels entails construction of the resolvent kernel in a form of a uniformly and absolutely converging series. Also, eigenfunctions, in view of their completeness and orthogonality, provide an efficient (due to exponential decay of eigenvalues) representation of the solution.

After discussing general spectral properties of the integral operator (Section 1.2), we propose original constructive techniques for obtaining asymptotical solution for the case of small (Section 1.3) and large size of interval (Section 1.4). When the interval is small, the integral equation can be approximated by another one which admits a commuting differential operator. This fact allows reduction of the problem to solving a second-order boundary-value problem whose solutions upon further approximations are prolate spheroidal harmonics (Slepian functions). When the interval is large, the problem can be transformed into an integro-differential equation on a shifted half-line. Integral kernel function of such problem consists of two terms: one depends on difference of the arguments, the other - on their sum. The latter turns out to be small for the large interval, and hence we arrive at an approximation by a convolution integro-differential equation which we solve by an extended Wiener-Hopf method. Connection of this half-line problem solution to the solution of the original equation inside the original interval is provided by analytic continuation that can be performed by means of solution of an elementary non-homogeneous ODE. Finally, we illustrate the obtained asymptotical results compare them with numerical solution (Section 1.5) and outline potential further work (Section 1.6).

1.2 General properties

Since the kernel $p_h(x)$ is an even and real-valued function, the operator $P_h\chi_A$ is self-adjoint, and because of the regularity of $p_h(x)$, the operator is also compact (e.g. as a Hilbert-Schmidt operator), and hence we have the standard spectral result [NS]

Theorem 1. *There exists $(\lambda_n)_{n=1}^{\infty} \in \mathbb{R}$, $\lambda_n \to 0$ as $n \to \infty$ and $(f_n)_{n=1}^{\infty}$ is a complete set in $L^2(A)$.*

Basic properties of eigenfunctions and eigenvalues can be outlined in the following proposition (see [Pon]).

Proposition 1. *For λ, f satisfying (1.1), the following statements hold true*
(a) All $(\lambda_n)_{n=1}^{\infty}$ are simple, and $\lambda_n \in (0,1)$,
(b) Each f_n is either even or odd, real-valued (up to a constant multiplicator), and $f_n \in C^{\infty}\left(\bar{A}\right)$. Moreover, $f_n(\pm a) \neq 0$.

The key result here is a non-vanishing behavior of eigenfunctions at the endpoints implying the multiplicity (simplicity) of the spectrum which, in particular, along with the evenness of p_h, further entails the real-valuedness and a certain parity of each solution f_n, a fact that will be used constructively in Section 1.4.

The upper bound for the eigenvalues in part (a) of Proposition 1 can be improved to

$$\lambda_n \leq \frac{2}{\pi} \arctan \frac{a}{h}, \quad n \in \mathbb{N}_+,$$

and asymptotically exponential decay of higher-order eigenvalues is given by

$$\log\lambda_n \simeq -n\pi\frac{K\left(\operatorname{sech}\left(\pi a/h\right)\right)}{K\left(\tanh\left(\pi a/h\right)\right)}, \quad n \gg 1 \tag{1.4}$$

where $K(x) := \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-x^2\sin^2\theta}}$ the complete elliptic integral of the first kind.

Note that, since the spectrum is simple, we can uniquely order eigenvalues as

$$0 < \cdots < \lambda_3 < \lambda_2 < \lambda_1 < 1,$$

and denote f_n the eigenfunction corresponding to λ_n, $n \in \mathbb{N}_+$. In what follows, when no comparison between different eigenvalues/eigenfunctions is made, we will continue writing simply f, λ instead of f_n, λ_n.

Finally, observe that a scaling argument (with a change of variable; see further) implies that the spectrum actually depends only on one parameter $\beta := h/a$. The main results will be formulated in terms of the magnitude of this parameter.

1.3 Small interval ($\beta \gg 1$)

Setting $\phi(x) := f(ax)$ for $x \in (-1,1)$, equation (1.1) rewrites as

$$\frac{\beta}{\pi}\int_{-1}^{1}\frac{\phi(t)}{(x-t)^2+\beta^2}dt = \lambda\phi(x), \quad x \in (-1,1), \tag{1.5}$$

Since eigenfunctions are defined up to a multiplicatory constant, for the sake of determinicity, let us choose this constant to be real and so that $\|\phi\|_{L^2(-1,1)} = 1$.

Observe that the kernel function $p_\beta(x)$ essentially coincides with $[0/2]$ Padé approximant of hyperbolic secant function

$$\operatorname{sech}(x) = \frac{1}{1+x^2/2} + \mathcal{O}\left(x^4\right), \quad |x| \ll 1,$$

hence the formulation (1.5) can be approximated by

$$\int_{-1}^{1}\operatorname{sech}\left((x-t)\sqrt{2}/\beta\right)\phi(t)\,dt = \pi\beta\lambda\phi(x) + \mathcal{O}\left(1/\beta^4\right), \quad x \in (-1,1), \tag{1.6}$$

and we therefore expect its solutions to be close to those of (1.5) for large β.

We drop the error term, postponing precise approximation error analysis to a further work, and now focus on an eigenvalue problem for the integral operator on the left of (1.6) which turns out to be again a positive compact self-adjoint operator on $L^2(-1,1)$ with simple spectrum and the same law of decay of eigenvalues (1.4). However, this seemingly more complicated integral operator has an advantage over the original one since it belongs to a rather unique family of convolution integral operators that admit a commuting differential operator [Gru, Wid]: eigenfunctions of an integral operator with the kernel $\frac{b\sin cx}{c\sinh bx}$

(with constants b, $c \in \mathbb{R} \cup i\mathbb{R}$) are also eigenfunctions of the differential operator $-\frac{d}{dx}\left(1-\frac{\sinh^2(bx)}{\sinh^2 b}\right)\frac{d}{dx}+\left(b^2+c^2\right)\frac{\sinh^2(bx)}{\sinh^2 b}$ with condition of finiteness at $x=\pm 1$. Therefore, taking $c=i\sqrt{2}/\beta$, $b=2\sqrt{2}/\beta$, and denoting $\frac{\mu}{2\sinh^2\left(2\sqrt{2}/\beta\right)}$ an eigenvalue of the differential operator, we reduce (1.6) to solving a boundary-value problem given by an ODE, for $x \in (-1,1)$,

$$\left(\left(\cosh\frac{4\sqrt{2}}{\beta}-\cosh\frac{4\sqrt{2}x}{\beta}\right)\phi'(x)\right)'+\left(\mu-\frac{6}{\beta^2}\left(\cosh\frac{4\sqrt{2}x}{\beta}-1\right)\right)\phi(x)=0 \tag{1.7}$$

and boundary conditions

$$\phi'(\pm 1)=\mp\frac{\mu+6/\beta^2\left(1-\cosh\left(4\sqrt{2}/\beta\right)\right)}{4\sqrt{2}\beta\sinh\left(4\sqrt{2}/\beta\right)}\phi(\pm 1). \tag{1.8}$$

Alternatively, introducing

$$\psi(s):=\frac{\phi\left(\frac{\beta}{2\sqrt{2}}\log\left[\left(e^{-2\sqrt{2}/\beta}-e^{2\sqrt{2}/\beta}\right)s+e^{-2\sqrt{2}/\beta}\right]\right)}{\left[\left(e^{-2\sqrt{2}/\beta}-e^{2\sqrt{2}/\beta}\right)s+e^{-2\sqrt{2}/\beta}\right]^{1/2}},$$

equation (1.6) can be brought into a simpler integral equation arising in context of singular-value analysis of the finite Laplace transform [BG]

$$\int_0^1\frac{\psi(t)}{s+t+\gamma}dt=-\pi\sqrt{2}\lambda\,\psi(s),\quad s\in(0,1),$$

with $\gamma:=2e^{-2\sqrt{2}/\beta}$. Here the operator in the left-hand side is a truncated Stieltjes transform which again, by commutation with a differential operator, can be reduced to solving an ODE, for $s \in (0,1)$,

$$\left(s(1-s)(\gamma+s)(\gamma+1+s)\psi'(s)\right)'-(2s(s+\gamma)+\mu)\psi(s)=0$$

with boundary conditions enforcing regularity of solutions at the endpoints

$$\psi'(0)=\frac{\mu}{\gamma(\gamma+1)}\psi(0),\quad \psi'(1)=-\frac{2(\gamma+1)+\mu}{(\gamma+1)(\gamma+2)}\psi(1).$$

Finally, it is remarkable that if we get back to (1.7) and Taylor-expand hyperbolic functions due to smallness of $1/\beta$, we obtain

$$\left(\left(1-x^2\right)\phi'(x)\right)'+\left(\mu-\frac{6}{\beta^2}x^2\right)\phi(x)=0,\quad x\in(-1,1), \tag{1.9}$$

an ODE that coincides with a well-studied equation [ORX, SP] whose solutions are bounded on $[-1,1]$ only for special values $\mu_n = \chi_n\left(\frac{\sqrt{6}}{\beta}\right)$, $n = \mathbb{N}_0$, and termed as prolate spheroidal (Slepian) wave functions $S_{0n}\left(\frac{\sqrt{6}}{\beta}, x\right)$ (with notation as in [SP]).

Note that even though differential operators presented here have the same eigenfunctions as integral ones, their eigenvalues are different. Once an eigenfunction ϕ_n is obtained, the corresponding eigenvalue of the original integral operator can be computed as $\lambda_n = \left\langle P_\beta\left[\chi_{(-1,1)}\phi_n\right], \phi_n\right\rangle_{L^2(-1,1)}$ (we use notation (1.2)).

1.4 Large interval ($\beta \ll 1$)

Let us set $\varphi(x) := f(xh)$ for $x \in (-a/h, a/h)$ and, by a change of variable, rewrite (1.1) as

$$\frac{1}{\pi}\int_{-1/\beta}^{1/\beta} \frac{\varphi(t)}{(x-t)^2+1}\,dt = \lambda\,\varphi(x), \quad x \in (-1/\beta, 1/\beta), \tag{1.10}$$

Denote $B := (-1/\beta, 1/\beta)$, choose normalization $\|\varphi\|_{L^2(B)} = 1$, and define the analytic continuation to $\mathbb{R}$ of the solution of (1.10) as

$$\varphi(x) = \frac{1}{\lambda\pi}\int_{-1/\beta}^{1/\beta} \frac{\varphi(t)}{(x-t)^2+1}\,dt. \tag{1.11}$$

Then, building up on a transformation introduced in [Gri], we can prove a non-evident yet very important result [Pon]

Lemma 1. *The analytic continuation of solution of* (1.10) *given by* (1.11) *satisfies*

$$\int_{\mathbb{R}\setminus B} R_0(x-t)\,\varphi(t)\,dt = \varphi(x), \quad x \in \mathbb{R}, \tag{1.12}$$

with

$$R_0(x) := -\frac{\sin(x\log\lambda)}{\tanh(\pi x)} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{n\lambda^n}{n^2+x^2}. \tag{1.13}$$

The parity of solutions (part (b) of Proposition 1) allows reducing an integration to only one half-line.

Theorem 2. *The analytic continuations $\varphi_{ext}(x) := \varphi(x+1/\beta)$ of even / odd solutions of* (1.10) *satisfy, for $x \in \mathbb{R}$,*

$$\int_0^\infty \left[R_0(x-t) \pm R_0\left(x+t+\frac{2}{\beta}\right)\right]\varphi_{ext}(t)\,dt = \varphi_{ext}(x), \tag{1.14}$$

as well as an integro-differential equation

$$\int_0^\infty \left[\mathscr{K}(x-t) \pm \mathscr{K}\left(x+t+\frac{2}{\beta}\right)\right]\varphi_{ext}(t)\,dt = \varphi''_{ext}(x) + \log^2\lambda\,\varphi_{ext}(x) \tag{1.15}$$

with the kernel function

$$\mathscr{K}(x) := -\left(\frac{d^2}{dx^2} + \log^2\lambda\right)\left(\frac{\sin(x\log\lambda)}{\tanh(\pi x)} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{n\lambda^n}{n^2+x^2}\right). \tag{1.16}$$

Here and onwards the upper sign corresponds to even solutions, the lower to odd ones.

Even though equation (1.14) (which is a direct rephrasing of (1.12)) is simpler than an integro-differential equation (1.15), it has a kernel (1.13) with an oscillatory behavior at infinity whereas $\mathscr{K}(x)$ decays. Indeed, it is easy to see that

$$R_0(x) \underset{|x|\gg 1}{\approx} \frac{\sin(x\log\lambda)}{\tanh(\pi x)} \approx \sin(|x|\log\lambda), \qquad \mathscr{K}(x) \underset{|x|\gg 1}{\approx} \frac{1}{x^2}.$$

This decaying property of the kernel function of (1.15) is crucial for construction of approximation on the right half-line region since the sum part of the kernel in (1.15) is uniformly small for $\beta \ll 1$ and $x,t > 0$: $\mathscr{K}(x+t+2/\beta) = \mathscr{O}(\beta^2)$. Neglecting this small term (and thus again postponing tedious error analysis to a further work), we end up with an equation of Wiener-Hopf type. Even though the presence of the derivative prohibits application of the standard Wiener-Hopf method, this difficulty can still be overcome by means of additional transformation leading to an explicitly solvable scalar Riemann-Hilbert problem giving thus an exact solution of the approximate equation. These results presented in greater detail in [Pon] are summarized here in Theorem 3 below. First of all, however, we should set up notations and define few auxiliary quantities, for $k \in \mathbb{R}$,

$$k_0 := -\frac{\log\lambda}{2\pi}, \quad \kappa := -\frac{\pi}{6} + 2k_0\log\left(e^{2\pi k_0} - 1\right) + \frac{1}{\pi}\mathrm{Li}_2\left(1 - e^{2\pi k_0}\right),$$

$$\hat{\mathscr{K}}(k) = \frac{2\pi^2\left(k_0^2 - k^2\right)e^{\pi(k_0-|k|)}}{\sinh(\pi(k_0-|k|))}, \quad G(k) := \frac{k^2-k_0^2}{2(k^2+1)}\left[1 + \coth(\pi(|k|-k_0))\right],$$

$$X_\pm(k) := \exp(P_\pm[\log G](k)) = G^{1/2}(k)\exp\left[\pm\frac{1}{2\pi i}\mathrm{p.v.}\int_{\mathbb{R}}\frac{\log G(\tau)}{\tau-k}d\tau\right],$$

$$\mathscr{C}(k) := \frac{(1+\kappa)\left(1+4\pi^2k_0^2\right)}{(1-2\pi ik)^2} - \frac{1-4\pi^2k_0^2+2\kappa}{1-2\pi ik} - P_+\left[\frac{2(1-\pi i\cdot)+\kappa}{(1-2\pi i\cdot)^2}\hat{\mathscr{K}}(\cdot)\right](k),$$

where we defined the Euler dilogarithm / Spence's function, Fourier transform and projection operators on spaces of analytic functions of upper and lower half-planes as follows

$$\mathrm{Li}_2(x) := -\int_0^x\frac{\log(1-t)}{t}dt = \sum_{n=1}^{\infty}\frac{x^n}{n^2}, \quad \hat{F}(k) := \mathscr{F}[F](x) = \int_{\mathbb{R}}F(x)e^{2\pi ikx}dx,$$

$$P_\pm[F](k) := \mathscr{F}\chi_{\mathbb{R}_\pm}\mathscr{F}^{-1}[F](k) = \frac{1}{2}F(k) \pm \frac{1}{2\pi i}\mathrm{p.v.}\int_{\mathbb{R}}\frac{F(t)}{t-k}dt.$$

Now we are ready to state the following

Theorem 3. *The integro-differential equation*

$$\int_0^\infty \mathscr{K}(x-t)\,\varphi_{ext}(t)\,dt = \varphi''_{ext}(x) + \log^2\lambda\,\varphi_{ext}(x), \qquad x>0, \tag{1.17}$$

possesses the unique solution given by

$$\varphi_{ext}(x) = \varphi\left(\frac{1}{\beta}\right)\left[e^{-x}(1+(1+\kappa)x) + \int_{\mathbb{R}} e^{-2\pi ikx}\frac{P_+[\mathscr{C}/X_-](k)}{4\pi^2(k^2+1)X_+(k)}dk\right]. \tag{1.18}$$

Moreover, this solution satisfies the endpoint condition $\varphi'_{ext}(0) = \kappa\varphi_{ext}(0)$.

Now we reuse Theorem 2 to recover solutions $\varphi(x) = \varphi_{ext}(x-1/\beta)$ inside the interval B due to the fact that the left-hand side of (1.15) is now computable from (1.18). This non-homogeneous ODE is easily solvable and depending on a choice of the sign in the integral term of (1.15) we obtain either even or odd family of solutions.

We conclude that even eigenfunctions are given by

$$\varphi(x)/\varphi\left(\frac{1}{\beta}\right) = C_1(\lambda,\beta)\cos(x\log\lambda) - \int_0^x N_0^+(t,\lambda,\beta)\sin((x-t)\log\lambda)\,dt, \tag{1.19}$$

and odd ones by

$$\varphi(x)/\varphi\left(\frac{1}{\beta}\right) = C_2(\lambda,\beta)\sin(x\log\lambda) - \int_0^x N_0^-(t,\lambda,\beta)\sin((x-t)\log\lambda)\,dt, \tag{1.20}$$

where

$$C_1(\lambda,\beta) := \frac{1}{\cos\left(\frac{1}{\beta}\log\lambda\right)}\left[1+\int_0^{\frac{1}{\beta}} N_0^+(t,\lambda,\beta)\sin\left(\left(\frac{1}{\beta}-t\right)\log\lambda\right)dt\right],$$

$$C_2(\lambda,\beta) := -\frac{1}{\sin\left(\frac{1}{\beta}\log\lambda\right)}\left[1+\int_0^{\frac{1}{\beta}} N_0^-(t,\lambda,\beta)\sin\left(\left(\frac{1}{\beta}-t\right)\log\lambda\right)dt\right],$$

$$N_0^\pm(x,\lambda,\beta) := \frac{1}{2\pi k_0}\int_0^\infty\left(\mathscr{K}\left(x-t-\frac{1}{\beta}\right)\pm\mathscr{K}\left(x+t+\frac{1}{\beta}\right)\right)\cdot \left[e^{-t}(1+(1+\kappa)t) + \int_{\mathbb{R}} e^{-2\pi ikt}\frac{P_+[\mathscr{C}/X_-](k)}{4\pi^2(k^2+1)X_+(k)}dk\right]dt.$$

Evaluation of derivatives and use of the boundary condition obtained in Theorem 3 lead to characteristic equations for even and odd eigenvalues, respectively,

$$\frac{\kappa}{\log\lambda}\cos\left(\frac{1}{\beta}\log\lambda\right)+\sin\left(\frac{1}{\beta}\log\lambda\right)=-\int_0^{\frac{1}{\beta}}N_0^+(t,\lambda,\beta)\cos(t\log\lambda)\,dt, \quad (1.21)$$

$$\frac{\kappa}{\log\lambda}\sin\left(\frac{1}{\beta}\log\lambda\right)-\cos\left(\frac{1}{\beta}\log\lambda\right)=-\int_0^{\frac{1}{\beta}}N_0^-(t,\lambda,\beta)\sin(t\log\lambda)\,dt. \quad (1.22)$$

1.5 Numerical illustrations

We verify our results of both Sections 1.3 and 1.4 by comparing them to a numerical collocation method applied to a rescaled formulation (1.5). We use a Gauss-Legendre quadrature rule with $N = 100$ points to approximate the integral operator

$$\sum_{j=1}^{N}\omega_j p_\beta(x-t_j)\,\phi_j=\lambda\phi(x), \quad x\in(-1,1) \quad (1.23)$$

with $\omega_j := \frac{2\left(1-t_j^2\right)}{N^2 P_{N-1}^2(t_j)}$, $P_{N-1}(x)$ being the $(N-1)$-th Legendre polynomial, $p_\beta(x)$ as in (1.3), and solve for $\phi_j := \phi(t_j)$, $j=1,\dots,N$, the following linear system

$$\sum_{j=1}^{N}p_\beta(t_i-t_j)\,\omega_j\phi_j=\lambda\phi_i, \quad i=1,\dots,N. \quad (1.24)$$

Eigenvalues are found from equating determinant of the system to zero, and continuous eigenfunctions are then reconstructed from (1.23) as

$$\phi(x)=\frac{1}{\lambda}\sum_{j=1}^{N}\omega_j p_\beta(x-t_j)\,\phi_j, \quad x\in(-1,1). \quad (1.25)$$

Numerical solutions demonstrate properties of a Sturm-Liouville sequence: even and odd eigenfunctions interlace and each ϕ_n, $n\in\mathbb{N}_+$, has exactly $n-1$ zeros.

In the case $\beta \gg 1$, we compare numerical results with prolate spheroidal wave functions which were computed using a Fortran code provided in [ZJ] and converted into a MATLAB program with the software f2matlab. We see on Figure 1.1 that even double approximation (first, by a cumbersome boundary-value problem (1.7)-(1.8) and then, proceeding further, by the one with ODE (1.9) for standard special functions) already furnishes excellent results.

In the case $\beta \ll 1$, we first solve characteristic equations (1.21)-(1.22) by finding intersection of curves in left- and right-hand sides as a function of $k_0 = -\frac{\log\lambda}{2\pi}$. They are plotted on Figure 1.2 along with vertical lines which correspond to eigenvalues obtained from the numerical solution described above. Plugging eigenvalues back in (1.19)-(1.20), we obtain even and odd family of solutions, respectively. We plot a couple of eigenfunctions on Figure 1.3, namely, the third even eigenfunction and the tenth odd. As on Figure 1.1, the asymptotic solutions are almost indistinguishable from the numerical, however, Figure 1.4 shows a breakdown of the asymptotic approximation for higher-order eigenfunctions (note also the discrepancies between abscisses of circled intersection points and vertical lines on Figure 1.2).

More plots of eigenfunctions and approximation errors are available in [Pon].

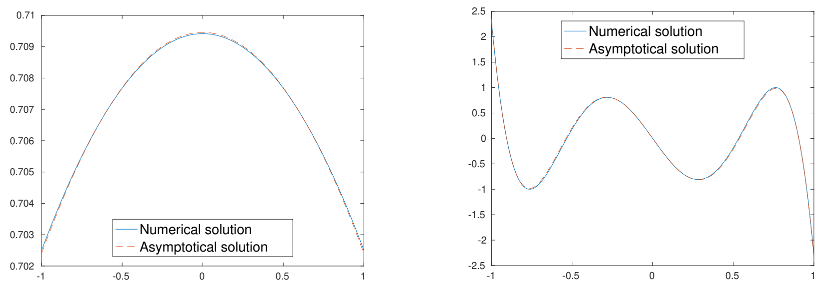

Fig. 1.1 Eigenfunctions ϕ_1 (left plot) and ϕ_6 (right plot). $\beta = 10$.

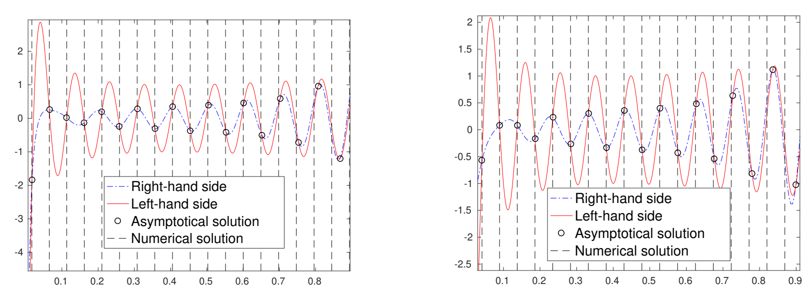

Fig. 1.2 Solving characteristic equations (1.21) (left plot) and (1.22) (right plot). $\beta = 0.1$.

1.6 Conclusion

We have presented two different methods to construct asymptotic solutions in cases when the interval is small and large. In the first case, we have exploited a rather specifique property of asymptotical closedness of the problem to an integral equation with an admissible commuting differential operator and concluded that solutions (eigenfunctions) can be approximated by those arising from either of two auxiliary Sturm-Liouville problems and, if further approximation is pursued, they

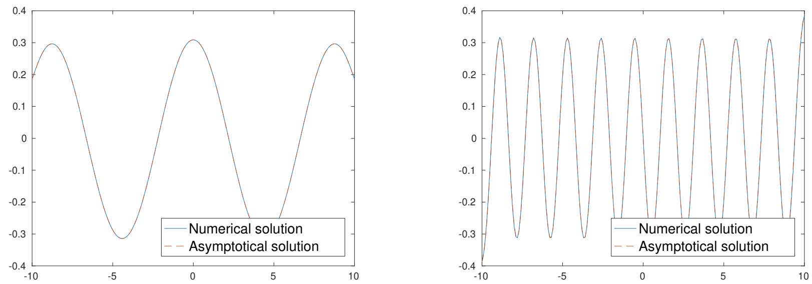

Fig. 1.3 Eigenfunctions φ_5 (left plot) and φ_{20} (right plot). $\beta = 0.1$.

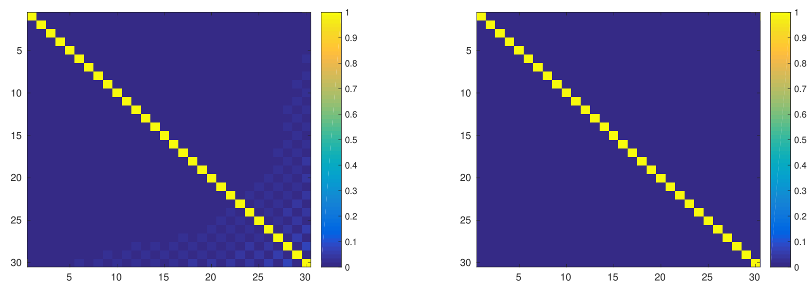

Fig. 1.4 Inner product matrices for solutions: asymptotic (left plot) and numerical (right plot). $\beta = 0.1$.

coincide with scaled versions of prolate spheroidal wave functions. In the second case, when the interval is large, the developed approach is rather general and should, in principle, be applicable to a wide class of integral equations with even kernels. Computational details (and simplicity of the form of a kernel for the integral equation on the half-line), however, will depend on analytical structure of the Fourier transform of a kernel. This is a natural topic for further investigation. Also, in the case of large interval, it is interesting to attempt to extend the results for $\lambda = -1$ (and a non-homogeneous term) recenlty obtained by Tracy and Widom [TW2, TW3] or those given by a boundary-layer type of asymptotic constructions in [AL], and compare these results with ours. Moreover, in the same large interval case, it was proven in [Pon] that equation (1.1) can be approximately reduced to a known non-homogeneous hypersingular equation known in air-foil theory p.v. $\int_{-a}^{a} \frac{f'(t)}{x-t} dt = \mu f(x) + g(x)$ which so far has been efficiently solved only numerically [KP, vT]. It seems worthy exploring this connection deeper on a constructive level. Nevertheless, of the primary importance is to provide rigorous justification of the obtained results (initiated in [Pon]) which were presented here heuristically and verified only numerically. This work in progress will soon be published in a forthcoming paper.

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