Assignment #5 solutions

March 31, 2011

Chapter 5 / Section 5 / Problem 2

For any $t \in \mathbb{R}$, define

$$\phi(t) := \|f + tg\|^2$$

Then

$$\phi(t) = (f + tg,\, f + tg) = \|g\|^2 t^2 + 2(f,g)t + \|f\|^2 \geq 0$$

Non-negativity of this expression implies that

$$\min_{t \in \mathbb{R}} \phi(t) \geq 0$$

Let us find the minimum explicitly.

$$\phi'(t) = 2\|g\|^2 t + 2(f,g) \qquad \Rightarrow \qquad t_0 = -\frac{(f,g)}{\|g\|^2}$$

$$\phi(t_0) = \frac{|(f,g)|^2}{\|g\|^2} - 2\frac{|(f,g)|^2}{\|g\|^2} + \|f\|^2 \geq 0 \qquad \Rightarrow \qquad |(f,g)|^2 \leq \|f\|^2 \cdot \|g\|^2$$

This furnishes the proof of the Schwarz inequality:

$$|(f,g)| \leq \|f\| \cdot \|g\|$$

Chapter 5 / Section 6 / Problem 5

$$\begin{cases} u_{tt}(x,t) & = c^2 u_{xx}(x,t) + e^t \sin(5x), \qquad 0 < x < \pi \\ u(0,t) & = u(\pi,t) = 0 \\ u(x,0) & = u_t(x,0) = \sin(3x) \end{cases}$$

We notice that the operator $\left(\partial_t^2 - c^2 \partial_x^2\right)$ doesn't change the form of non-homogeneous term $e^t \sin(5x)$, therefore this term can be "killed" by an appropriate shift function.

Let

$$u(x,t) = v(x,t) + A \cdot \left(e^t \sin(5x)\right)$$

where constant A is to be found from the condition on equation for $v(x,t)$ to be homogeneous.

Compute

$$u_{tt} = v_{tt} + A \cdot e^t \sin(5x)$$

$$u_{xx} = u_{xx} - 25A \cdot e^t \sin(5x)$$

Subbing this into the original PDE, we obtain

$$v_{tt}(x,t) = c^2 v_{xx}(x,t) + e^t \sin(5x) \underbrace{\left[1 - \left(25c^2 + 1\right) A\right]}_{=0}$$

Therefore,

$$A = \frac{1}{25c^2 + 1}$$

As one can see, the problem for $v(x,t)$ also inherits boundary conditions of the original problem, and hence

$$\begin{cases} v_{tt}(x,t) & = c^2 v_{xx}(x,t), \qquad 0 < x < \pi \\ v(0,t) & = v(\pi,t) = 0 \\ v(x,0) & = -A \cdot \sin(5x); \quad v_t(x,0) = \sin(3x) - A \cdot \sin(5x) \end{cases}$$

This is homogeneous and thus familiar to us problem which has the general solution

$$v(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(nct) + B_n \sin(nct)\right] \sin(nx)$$

From initial conditions:

$$v(x,0)=\sum_{n=1}^{\infty} A_n \sin{(nx)} \qquad \Rightarrow \qquad A_n = 0 \ \ (\forall n \neq 5)\,,\ A_5=-A$$

$$v_t(x,0)=\sum_{n=1}^{\infty} B_n \cdot n \cdot c \cdot \sin{(nx)} \qquad \Rightarrow \qquad B_n = 0 \ \ (\forall n \neq 3,\, 5)\,,\ B_3=\frac{1}{3c},\ B_5=-\frac{A}{5c}$$

Plugging these into the general solution above, we obtain

$$v(x,t)=\frac{1}{3c}\sin{(3ct)}\sin{(3x)}-\frac{1}{25c^2+1}\left[\cos{(5ct)}+\frac{1}{5c}\sin{(5ct)}\right]\sin{(5x)}$$

Now we get back to the original problem and write the final solution

$$u(x,t)=\frac{1}{3c}\sin{(3ct)}\sin{(3x)}-\frac{1}{25c^2+1}\left[\cos{(5ct)}+\frac{1}{5c}\sin{(5ct)}-e^t\right]\sin{(5x)}$$

Chapter 6 / Section 1 / Problem 9

$$\begin{cases} \Delta u = 0, & 1<r<2 \\ u\,|_{r=1} = 100 & \\ \frac{\partial u}{\partial r}\,|_{r=2} = -\gamma & \end{cases}$$

a)

The fact that the boundary conditions are spherically symmetric (don't depend on angles) suggests the solution to possess spherical symmetry as well:

$$u(r,\theta,\phi)=u(r)$$

Then the PDE becomes an ODE that can be integrated by separation of variables (or by introducing $v(r)=r\cdot u(r)$ which solves $v''=0$, as we did in class)

$$u''+\frac{2}{r}u'=0 \qquad \Rightarrow \qquad \underbrace{\frac{u''}{u'}}_{=(\log u')'}=-\frac{2}{r} \qquad \Rightarrow \qquad \log u' = \underbrace{-2\log r+\log C_1}_{=\log(C_1/r^2)}$$

$$u' = \frac{C_1}{r^2} \qquad \Rightarrow \qquad u(r) = -\frac{C_1}{r} + C_2$$

From boundary conditions we find

$$u(1) = 100 \qquad \Rightarrow \qquad C_2 = C_1 + 100$$

$$u'(2) = -\gamma \qquad \Rightarrow \qquad \frac{C_1}{4} = -\gamma$$

Hence

$$C_1 = -4\gamma, \qquad C_2 = 100 - 4\gamma$$

$$u(r) = \frac{4\gamma}{r} + 100 - 4\gamma$$

b)

Since we have found the explicit solution, we see that it is monotonically decreasing function of radial variable r attaining its maximal and minimal values on inner and outward boundaries respectively, which is in perfect agreement with maximum/minimum principle for a harmonic function:

$$u(1) = 100, \qquad u(2) = 100 - 2\gamma$$

c)

From the previous line it follows

$$u(2) = 20 \qquad \Rightarrow \qquad 100 - 2\gamma = 20 \qquad \Rightarrow \qquad \gamma = 40$$