Assignment #2 solutions

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Chapter 2 / Section 1 / Problem 10

$$u_{xx} + u_{xt} - 20u_{tt} = 0, \qquad u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x)$$

We start with factorization of the differential operator:

$(\partial_x + a\partial_t)(\partial_x + b\partial_t)u = 0 \qquad \Leftrightarrow \qquad u_{xx} + (a+b)u_{xt} + abu_{tt} = 0$

Hence $a + b = 1$, $ab = -20$. Take $a = 4$, $b = -5$.

We want to change coordinates to (ξ, η) such that

$$\begin{cases} \partial_\xi = \underbrace{1}_{=\partial x/\partial \xi} \cdot \partial_x + \underbrace{4}_{=\partial t/\partial \xi} \cdot \partial_t \\ \partial_\eta = \underbrace{1}_{=\partial x/\partial \eta} \cdot \partial_x + \underbrace{(-5)}_{=\partial t/\partial \eta} \cdot \partial_t \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = \xi + \eta \\ t = 4\xi - 5\eta \end{cases} \quad \Leftrightarrow \quad \begin{cases} \xi = (5x + t)/9 \\ \eta = (4x - t)/9 \end{cases}$$

Then the equation becomes very easy to solve:

$$u_{\xi\eta} = 0 \quad \Leftrightarrow \quad u(\xi, \eta) = F_0(\xi) + G_0(\eta) \quad \Leftrightarrow \quad u(x,t) = \underbrace{F_0\left(\frac{5}{9}(x + t/5)\right)}_{:=F(x+t/5)} + \underbrace{G_0\left(\frac{4}{9}(x - t/4)\right)}_{:=G(x-t/4)}$$

Hence the general solution is

$$u(x,t) = F(x + t/5) + G(x - t/4), \tag{1}$$

where F, G arbitrary differentiable functions to be found from the initial data:

$$\begin{cases} u(x,0) = F(x) + G(x) = \phi(x) \\ u_t(x,0) = \frac{1}{5}F'(x) - \frac{1}{4}G'(x) = \psi(x) \end{cases} \quad \Rightarrow \quad \begin{cases} F(x) + G(x) = \phi(x) \\ \frac{1}{5}F(x) - \frac{1}{4}G(x) = \int_{s_0}^{x} \psi(s)ds \end{cases},$$

where we don't write the constant of integration due to arbitrariness of s_0. Thus

$$\begin{cases} F(x) = \frac{5}{9}\left[\phi(x) + 4\int_{s_0}^{x}\psi(s)ds\right] \\ G(x) = \frac{4}{9}\left[\phi(x) - 5\int_{s_0}^{x}\psi(s)ds\right] \end{cases}$$

Plugging this back into the general solution (1), we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{9}\left[5\phi(x+t/5) + 4\phi(x-t/4) + 20\left\{\int_{s_0}^{x+t/5}\psi(s)ds - \int_{s_0}^{x-t/4}\psi(s)ds\right\}\right] = \\ &= \frac{1}{9}\left[5\phi(x+t/5) + 4\phi(x-t/4) + 20\int_{x-t/4}^{x+t/5}\psi(s)ds\right] \end{aligned}$$

Chapter 2 / Section 2 / Problem 3

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) \tag{2}$$

a)

Let $v(x,t) := u(x-y,\,t)$. Then we compute $v_{tt}(x,t) = u_{tt}(x-y,\,t)$, $v_{xx}(x,t) = u_{xx}(x-y,\,t)\cdot\underbrace{\left[\frac{d(x-y)}{dx}\right]^2}_{=1}$.

But evaluating derivatives in (2) at $x = x-y$, we obtain

$$u_{tt}(x-y,\,t) = c^2 u_{xx}(x-y,\,t) \qquad \Rightarrow \qquad v_{tt}(x,t) = c^2 v_{xx}(x,t)$$

Hence we conclude that $v(x,t) = u(x-y,\,t)$ solves the equation (2).

b)

Let us differentiate (2) with respect to x:

$$u_{ttx}(x,t) = c^2 u_{xxx}(x,t)$$

Now according to the Clairaut / Schwarz theorem, under assumption that u has continuous partial derivatives of the third order, we can interchange the order of differentiation: $u_{ttx}(x,t) = u_{xtt}(x,t)$. Then

$$(u_x(x,t)) = c^2\left(u_x(x,t)\right)_{xx}$$

Thus $u_x(x,t)$ satisfies the wave equation (2).

c)

Set $v(x,t) := u(x-y,\, t)$. Straightforward computations yield: $v_{tt}(x,t) = a^2 \cdot u_{tt}(ax,\, at)$, $v_{xx}(x,t) = a^2 \cdot u_{xx}(ax,\, at)$.

Evaluating derivatives in (2) at $x = ax$, $y = ay$, we arrive at

$$u_{tt}(ax,\, at) = c^2 u_{xx}(ax,\, at) \quad \underset{\text{multiplying both sides by } a^2}{\Rightarrow} \quad v_{tt}(x,t) = c^2 v_{xx}(x,t)$$

Therefore $v(x,t) = u(ax,\, at)$ solves the equation (2).

Chapter 2 / Section 3 / Problem 6

Let $u(x,t)$, $v(x,t)$ be such that

$$\begin{cases} u_t(x,t) = k u_{xx}(x,t), & 0 < x < l,\, t > 0 \\ u(0,t) = f_1(t),\, u(l,t) = f_2(t), & t > 0 \\ u(x,0) = \phi(x), & 0 \le x \le l \end{cases}, \qquad \begin{cases} v_t(x,t) = k v_{xx}(x,t), & 0 < x < l,\, t > 0 \\ v(0,t) = g_1(t),\, v(l,t) = g_2(t), & t > 0 \\ v(x,0) = \psi(x), & 0 \le x \le l \end{cases}$$

Introduce $w(x,t) := u(x,t) - v(x,t)$. By linearity, it solves the following problem

$$\begin{cases} w_t(x,t) = k w_{xx}(x,t), & 0 < x < l,\, t > 0 \\ w(0,t) = f_1(t) - g_1(t),\, w(l,t) = f_2(t) - g_2(t), & t > 0 \\ w(x,0) = \phi(x) - \psi(x) & 0 \le x \le l \end{cases}$$

Since we know that $u(0,t) \le v(0,t)$, $u(l,t) \le v(l,t)$, $u(x,0) \le v(x,0)$, we immediately have

$$\begin{cases} w(0,t) = f_1(t) - g_1(t) \le 0 \\ w(l,t) = f_2(t) - g_2(t) \le 0 \\ w(x,0) = \phi(x) - \psi(x) \le 0 \end{cases}$$

Therefore, applying maximum principle to $w(x,t)$, we obtain

$$w(x,t) \le 0,\ 0 \le x \le l,\ t \ge 0,$$

that is $u(x,t) \le v(x,t)$ for $0 \le x \le l$, $t \ge 0$.